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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Annual Meeting, Richmond and Williamsburg, Va., December 27–31, 1938.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1938 and reported to the Secretary.

ALLEGHENY MOUNTAIN, May 14.
ILLINOIS, Carbondale, May 13–14.
INDIANA, Terre Haute, May 6–7.
IOWA, Sioux City, April 15–16.
KANSAS, Pittsburg, April 2.
KENTUCKY, Morehead, May 14.
LOUISIANA-MISSISSIPPI, Starkville, Miss., March 11–12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 7; College Park, Md., December 10.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, Collegeville, May 14; Minneapolis, October 28
MISSOURI, Rolla, April 23.
NEBRASKA, Hastings, May 6.
OHIO, Columbus, March 31.
OKLAHOMA, Oklahoma City, February 11.
PHILADELPHIA, Collegeville, Pa., Nov. 26.
ROCKY MOUNTAIN, Boulder, Colo., April 15–16.
SOUTHEASTERN, Atlanta, Ga., April 1–2.
SOUTHERN CALIFORNIA, Claremont, March 26.
SOUTHWESTERN, Albuquerque, N.M., April 25–26.
TEXAS, Fort Worth, April 22.
WISCONSIN, West De Pere, May 14.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE MAY MEETING OF THE ILLINOIS SECTION

The nineteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at the Southern Illinois State Normal University, Carbondale, on Friday and Saturday, May 13–14, 1938. The chairman of the Section, Professor W. B. Storm, presided.

The attendance was about sixty, including the following thirty members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. G. Ayre, Walter Bartky, Laura E. Christman, C. E. Comstock, D. R. Curtiss, A. E. Gault, M. C. Hartley, E. W. Hellmich, E. C. Kiefer, Ruth G. Mason, J. R. Mayor, E. B. Miller, E. J. Moulton, Mary W. Newson, Echo D. Pepper, E. W. Ploenges, W. C. Randels, R. G. Sanger, H. A. Simmons, Norma K. Stelford, Eugene Stephens, Jessica Y. Stephens, R. C. Stephens, W. B. Storm, C. J. Stowell, M. E. Wescott, F. E. Wood, Alice K. Wright.

There were program sessions on Friday afternoon and Saturday morning. After a dinner Friday evening at beautiful Midland Hills Country Club the group adjourned to the Little Theatre on the campus and were entertained by two movies: "Parabola," a sound film designed and directed by Mr. Rutherford Boyd, showing mathematical design in art; and "Astronomy movies" planned and directed by Professor Walter Bartky of the University of Chicago. Professor Bartky was present and gave a short explanatory talk preceding the presentation of the film.

For the year 1938–39, Professor J. R. Mayor of Southern Illinois State Normal University was elected chairman; the department of mathematics of Knox College was asked to select one of its members for vice-chairman; and Professor C. N. Mills of Illinois State Normal University was elected secretary-treasurer. The next meeting will be held at Knox College, May 12–13, 1939.

The following papers were presented:

1. "Analytic geometry with complex coördinates and applications" by Professor F. E. Wood, Northwestern University.

2. "Some dissection problems" by Dr. Ruth G. Mason, University of Illinois.

3. "Applied operational mathematics" by Professor Eugene Stephens, Washington University.

4. "Teaching mathematical analysis in the junior college" by Dr. J. S. Georges, Wright Junior College.

5. "An application of the theory of summability" by Professor H. L. Garabedian, Northwestern University, introduced by Professor Curtiss.

6. "Some interesting problems in the study of porisms" by Dr. Josephine H. Chanler, University of Illinois, introduced by Dr. Echo D. Pepper.

7. "Relative claims of special education and general education upon the mathematics courses of the high school and junior college." A panel discussion by Dr. R. G. Sanger, University of Chicago, Professor Mabel Heren, Knox College, Professor E. W. Hellmich, Northern Illinois State Teachers College,

and E. B. Hexter, Belleville Township High School; Professor J. R. Mayor, Southern Illinois State Normal University, Chairman.

In the absence of the authors the papers by Dr. Georges and by Professor Heren were read by title only.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The basic formulas of complex (or circular) coördinates were developed in a series of simple, logical steps by Professor Wood; also an attempt was made to show how these coördinates have been applied to metric theorems regarding triangles and their related points, lines, and circles.

2. Dr. Mason defined the solution of a dissection problem to be a construction which divides a given figure or figures into parts which by euclidean transformations can be converted into a pre-assigned figure or figures. Methods were shown for cutting up two squares so that the pieces would form one square, for transforming any rectangle into a square, and for separating a square into four pieces that can be assembled into an equilateral triangle. Dudeney's discussion of the Greek Cross was generalized for the stepagon. The step process was explained and the possibility of so-called paradoxes was illustrated by the geometrical "proof" that $441=442$.

3. Professor Stephens gave definitions for operations with permissible algebraic transformations and identities by the method of abstraction of operators. He included the operator D, the partial operator and its generalization, and the noncommutative operator XD. The uses and applications of these then followed, with references to the history and literature covering over one hundred years.

4. To meet the needs of students who seek a general education and those who prepare for professional schools—the chief group in a public Junior College—Dr. Georges reported that the freshman course in mathematics at Wright Junior College includes materials of college algebra, trigonometry, analytic geometry, and elements of calculus, stressing the idea of mathematical analysis, and presenting mathematics as a method of thinking.

5. Professor Garabedian discussed briefly applications of the theory of summability to a problem in the flow of heat in the domain of mathematical physics. The principal supporting theorems are classical ones due to Fejer and Bromwich. The application is due to C. N. Moore of the University of Cincinnati. The speaker has given the shortest and most recent proof of Bromwich's theorem and has published results on applications of this theorem.

6. In her paper Dr. Chanler defined a porism as a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of an infinite number of solutions. Illuminating examples were presented from elementary geometry, including the classic theorem of Poncelet in 1822. Modern illustrations and discussions of the Poncelet theory were taken from the literature, including some which relate the poristic polygons to the theory of the elliptic cubic. Algebraic generalization leads to the poristic double binary form $F_{k,K}\equiv(\alpha t)^k(aT)^K$, investigated by A. B. Coble. Intimately

related to certain poristic forms are the involution curves like the Lüroth quartic curve. Examples of poristic double binary forms and involution curves were given from the writer's papers on these subjects.

7. In this panel discussion Dr. Sanger emphasized the need of requiring mathematics in the high school, but not of all college students. He indicated how a plan of giving a choice of courses, based on the need for technical knowledge, was carried out at the University of Chicago.

Professor Hellmich defended required mathematics courses in high school and college and presented a plan for three years of required mathematics, two of which would come in the high school, and one in college. Rather than defending the traditional courses in mathematics in both high school and college, a broad outline of work was presented for the three years which he thought much better adapted to the claims of general education.

Mr. Hexter suggested that criticisms of high school teaching by college teachers are reflections of the type of work these professors are doing. He raised some questions about the preparation of college teachers for teaching and mentioned the possible desirability of high school experience. A defense was given for trigonometry and college algebra in the high school.

EDITH I. ATKIN, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Thursday, March 31, 1938, with an afternoon session, a dinner, and an evening session. Professor O. E. Brown, chairman of the Section, presided at these sessions. The Section was happy to have Professor Karl Menger of the University of Notre Dame as guest-speaker.

Ninety-two persons registered attendance, fifty-four of whom were members of the Association, namely: W. E. Anderson, Max Astrachan, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, O. E. Brown, R. S. Burington, I. W. Burr, W. D. Cairns, F. E. Carr, E. H. Clarke, Rufus Crane, O. L. Dustheimer, T. M. Focke, N. A. Gilbert, B. C. Glover, R. C. Hildner, E. J. Hirschler, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, Florentina Mathias, Karl Menger, C. C. Morris, Max Morris, J. R. Musselman, J. R. Overman, W. A. Patterson, Jesse Pierce, H. S. Pollard, S. E. Rasor, C. E. Rhodes, R. F. Rinehart, N. S. Risley, S. A. Rowland, W. G. Simon, Mary E. Sinclair, H. E. Stelson, Otto Szász, C. F. Thomas, Fern Welker, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, J. B. Winslow, C. R. Wylie, Jr., B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, C. O. Williamson, College of Wooster; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, C. E. Rhodes, Ohio State University; Member of Program Committee, H. A. Bender, University of

Akron. It is expected that the next meeting will be held on Thursday, April 6, 1939, at the Ohio State University.

The following papers were read:

1. "The application of determinants and projective transformations to nomography" by the chairman of the Section, Professor O. E. Brown, Case School of Applied Science.

2. "The line of images" by Professor J. R. Musselman, Western Reserve University.

3. "A series of line involutions in S_3 defined as point transformations of a V_4^2 of S_5 into itself" by Dr. C. R. Wylie, Jr., Ohio State University.

4. "The standard deviation of the medians of small samples" by Professor H. S. Pollard, Miami University.

5. "A note on the elementary symmetric functions" by H. Reingold, University of Cincinnati, introduced by Professor Barnett.

6. "Solutions of $dx/dt=(1/2x)\sum_{h=0}^{\infty} f_h(t)x^h$, $f_0(t)\neq 0$, in the vicinity of branch points, in terms of infinite series of definite integrals" by Professor Jesse Pierce, Heidelberg College.

7. "The foundations of projective and affine geometry" by Professor Karl Menger, University of Notre Dame, by invitation.

8. A symposium, "The mathematician in action" by Professors C. C. Morris, B. F. Yanney, and Karl Menger.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Brown stated the fundamental problem of alignment chart design, namely, given a relation, $F(u, v, w)=0$ in three variables u, v, w, to determine three curves C_u, C_v, C_w, and graduations upon them, such that the graduations corresponding to any three particular values of those variables will lie on a straight line if and only if those particular values satisfy the given relation. He illustrated by a number of examples, the role played by determinants of the third order in solving this problem, and the possible variation of the solution through projective transformations.

2. This paper appears in this issue of the MONTHLY.

3. A family of lines in five dimensions, with the property that a unique line of the system passes through any given point, can be used to map a quadratic primal into itself. If the primal be the one which represents the lines of three dimensions, such a mapping relation defines a line involution in three dimensions. Dr. Wylie considered five systems of lines with the above property; namely, the lines through a fixed point, the lines meeting a fixed line and a fixed S_3, the lines meeting two fixed planes, the lines meeting a fixed plane and two fixed S_3's, and the lines meeting four fixed S_3's. The associated line involutions in three dimensions range in order from one to seven, and include a number of well known transformations as special cases.

4. Professor Pollard showed that the formula which is ordinarily used to calculate the standard deviation of the median yields only an approximation to its

actual value, and that for samples which do not contain a fairly large number of items, and for distributions in which the median is located at a point of relatively small frequency, this approximation may be untrustworthy. He proposed another method of measuring the stability of the median, which involved the determination of the frequency function according to which the medians of samples drawn from a given distribution are distributed, and the calculation of the third quartile of this frequency distribution of medians. The deviation of this third quartile from the median of the distribution was used as a measure of the probable error of the median.

5. Mr. Reingold discussed an interesting property of the elementary symmetric functions $E_1, E_2, \cdots, E_n$ of the independent variables $x_1, x_2, \cdots, x_n$. He showed that the Jacobian of the elementary symmetric functions is, except possibly for sign, equal to the Vandermonde determinant of the variables $x_1, x_2, \cdots, x_n$; also that the Jacobian of the n coefficients of the characteristic equation of the nth order square matrix a_{ij}, with respect to any set of n elements $a_{r_1s_1}, a_{r_2s_2}, \cdots, a_{r_ns_n}$, of the matrix is, except possibly for sign, equal to a generalized determinant of Vandermonde $\left| a_{s_1r_1}^{(i-1)} a_{s_2r_2}^{(i-1)}, \cdots, a_{s_nr_n}^{(i-1)} \right|$, $(i=1, 2, \cdots, n)$, where a determinant of nth order is written by exhibiting its ith row.

6. Professor Pierce discussed solutions of a differential equation of the first order and first degree with a pole of order one with respect to the dependent variable. Two solutions with the initial conditions $x(t_0)=0$ are found in terms of infinite series of definite integrals. The method of finding the formal solution functions is similar to that used by the author in finding solutions of systems of linear differential equations in the vicinity of singular points (this MONTHLY, November, 1936) and in finding solutions of systems of differential equations in terms of infinite series of definite integrals (*Duke Mathematical Journal*, December, 1937).

7. Professor Menger gave a brief exposition of the foundation of geometry on a few simple assumptions concerning two operations in a class of undefined elements, the operations corresponding to joining and intersecting of the linear elements of space. The theory may be called *algebra of geometry* on account of its analogy to algebra of numbers, based in its modern abstract form on assumptions concerning two operations in a class of undefined elements, corresponding to addition and multiplication of numbers; and to algebra of classes which likewise has been developed as a theory of two operations, corresponding to joining and intersecting of classes. (This latter theory can be obtained as a special case of algebra of geometry.) From the above mentioned basis we can derive projective, affine, and non-euclidean geometry, each one by adding a specific assumption concerning parallelism.

8. Professor Morris spoke of the mathematician in business, and in particular analyzed the mathematical aspects of the recovery program of President Roosevelt, showing why he took the steps he did, their result, and prophesying what his future steps will be. Professor Yanney spoke of the mathematician as a teacher, touching lightly on his qualifications, his opportunities for adventure

and pleasure in every new class, the flexibility of his methods, and incidentally, his strategic position today in our educational system. Professor Menger spoke of the mathematician in research, pointing out the stimulating influence of teaching activity upon research. Questions which arise from reading, lecturing, or from intelligent students should be answered before one's attention weakens or shifts. Six months might well be a limit beyond which one must not postpone attack.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-seventh regular meeting of the Iowa Section of the Mathematical Association of America was held at Morningside College, Sioux City, Iowa, on Friday and Saturday, April 15–16, 1938, in conjunction with the fifty-second regular meeting of the Iowa Academy of Science. The chairman, Professor L. E. Ward, presided.

The attendance was about twenty-seven, including the following fourteen members of the Association: W. E. Ekman, Cornelius Gouwens, M. E. Graber, Dora E. Kearney, R. B. McClenon, E. E. Moots, E. N. Oberg, H. L. Rietz, Fred Robertson, W. J. Rusk, E. R. Smith, L. E. Ward, C. W. Wester, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the Section elected for 1938–39 are as follows: Chairman, E. E. Moots, Cornell College; Vice-Chairman, A. T. Craig, University of Iowa; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by their host, Morningside College, was adopted at the business session. The invited address was given by Professor R. B. McClenon of Grinnell College. The following seventeen papers were read:

1. "Osculatory interpolation formulas for finding approximate values of definite integrals" by Professor J. F. Reilly, State University of Iowa, by title.
2. "Generalization of the formula of Leibniz" by Fred Robertson, Iowa State College.
3. "What is a number?" by Professor C. W. Wester, Iowa State Teachers College.
4. "Note on the Gamma function" by Professor W. J. Rusk, Grinnell College.
5. "Some properties of a class of polynomials associated with Bateman's k-function" by J. M. Bates, Iowa State College, introduced by Professor E. R. Smith.
6. "The semi-invariants of Thiele of a binomial distribution" by Professor A. T. Craig, State University of Iowa, by title.

7. "Lagrange and his influence on modern mathematics" by Professor R. B. McClenon, Grinnell College.

8. "Periodogram analysis of selected agricultural prices" by R. L. Anderson, Iowa State College, introduced by the Secretary.

9. "An approximate solution by a generalized method for a square clamped plate" by C. J. Thorne, Iowa State College, introduced by the Secretary.

10. "Some problems dealing with the flow in rivers" by Professor E. E. Moots, Cornell College.

11. "Some properties of the orthic triangle" by Professor Helen K. Milleson, Buena Vista College, introduced by Professor Roscoe Woods.

12. "On the 'Curve of Deaths' and the associated 'Curve of Lives' " by Professor H. L. Rietz, State University of Iowa.

13. "An application of trilinear coördinates to some simple boundary value problems" by Professor D. L. Holl, Iowa State College, introduced by the Secretary.

14. "A rule for solving integral equations having special kernel functions" by Professor E. N. Oberg, State University of Iowa.

15. "Hysteretic probability" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

16. "Mathematical rigor and validity" by Professor M. E. Graber, Morningside College.

17. "An asymptotic formula for the small non-vanishing zeros of confluent hypergeometric functions" by Professor E. R. Smith, Iowa State College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

2. J. A. Joseph in the January 1938 issue of this MONTHLY shows that the theorem of Leibniz

$$D^n(u\cdot v) = \sum_i \binom{n}{i} D^i u D^{n-i} v,$$

known to be valid for positive integral values of n, holds for negative integral values. In the present paper Mr. Robertson showed by the Bourlet method that the theorem is valid for all real values of the index n.

3. To clear up the confusion and mysticism resulting from trying to think number as cardinal number, and of a cardinal number as a collection of objects or else as a symbol for the collection, Professor Wester stated that a simple definition of number is needed. It should be authoritative to serve as a guide to teachers, textbook makers, and school authorities generally. Such definitions are now possible though manifestly impossible before the researches of the last century.

4. First by using the trigonometrical form of the integral for $B(m, n)$ and its expression in terms of Γ-functions and transforming the integrand function by multiplication by $(\sin^2\theta+\cos^2\theta)^p$, and expanding the new integrand function, Professor Rusk was able to express it as a summation in B-functions or Γ-func-

tions which were identities in p for all positive integral values of p. Secondly by replacing $\sin^{2p}\theta$ by $(1-\cos^2\theta)^p$ and expanding he was able to get analogous formulas which hold so long as m, n, and p are greater than zero. Then by using the formula $(n+k)=(n+k-1)_k\Gamma(n)$ he deduced formulas which might be considered analogous to Vandermonde's Theorem when the p was a positive integer, and as generalizations when the m, n, and p are positive but not integers.

5. The polynomials discussed by Mr. Bates are associated with Bateman's k-function which is a special type of Whittaker's function $W_{k,m}(z)$, corresponding to the case $m=1/2$. The family of polynomials is defined by the generating function

$$e^{2sx/(1+s)} = \sum_{n=0}^{\infty} u_n(x)s^n,$$

where $u_n(x)$ is the nth polynomial. A differential form, an integral form, difference and differential equations, and orthogonal relations were derived. Also two expansions for the zeros of the polynomials were obtained in terms of the zeros of the Bessel functions.

6. In this note, Professor Craig showed that $\lambda_{k+1}=pqd\lambda_k/dp$ is a recursion formula for the semi-invariants of Thiele of a binomial distribution where $\lambda_1=N_p$, p being the probability of occurrence of an event in a single trial, $q=1-p$, and N the number of trials. An explicit formula for λ_k was also found.

7. Professor McClenon outlined the three periods in the life of Lagrange, the Turin, Berlin, and Paris periods, and quoted from letters and other sources to show something of the circumstances under which Lagrange did his work. He also outlined some of the most outstanding of Lagrange's contributions to mathematics, and showed how many fields are making use of ideas which go back to Lagrange.

8. The periodogram analysis of Professor Arthur Schuster was used by Mr. Anderson to study the periodicity of cyclical variation in wholesale commodity prices for the period 1890–1937. If there is a regular period, P, in the variation, this variation can be represented by a Fourier series, but only the first harmonic term, $R_1 \sin(2\pi x/p+\phi_1)$, is considered. Rye prices disclosed a marked period of 39 months in the cyclical variation, rubber disclosed no true period, and hides revealed a true period of 42 months for only the pre-war data.

9. The generalization may be illustrated by a boundary value problem in which the Ritz and Boussinesq methods are made special cases of a more general functional method. This involves the use of a functional in the determination of the coefficients of the expansion function. Mr. Thorne applied the method to a homogeneous, square, clamped plate with a center point load. The result obtained (1) yields a center deflection of $.02245\ Pa^2/N$ in close agreement with the work of Barta and Marcus; (2) satisfies the differential equation and boundary equations at all points; and (3) furnishes a line integral of boundary shear yielding a value .003% high.

10. If we introduce into a river flowing at low water stage an additional

volume of water, a swell or flood wave is produced and propagated in a certain manner. A simple flood wave is a long, positive wave superposed upon a river in steady flow. After developing the usual equations of continuity and motion Professor Moots showed that the velocity of a small flood wave when treated as the rate of movement of a constant flow, is equivalent to 1.5 times the mean channel velocity. The differential equation of the monoclinal or rising flood wave was next developed and integrated. Profiles, plotted for a 12-foot floodstage and an 8-foot normal flow, with river slopes ranging from .002 to .0004, show a variation of effective wave lengths ranging from 19.2 miles to 96 miles. This wave, rather often, closely characterizes the flow in large rivers during certain flood stages.

11. Professor Milleson presented an enumeration of triangles similar to the pedal triangle of the orthocenter, in an original arrangement of relations existing between the orthic triangle and various systems of circles, especially the Miquel circles and two coaxaloid systems: the Tucker circles, derived from coaxal circles through the Brocard points of the triangle of reference, and another system, of which the circumcircle is also a member, whose radii are changed by the constant multiplier 1/2 and whose radical axis is perpendicular to the Euler line.

12. In this paper Professor Rietz introduced a set of concepts in a brief and simple manner leading up to the use of continuous biometric functions in the characterization of mortality as a function of age. He followed this by an exposition of some of the interesting properties of the "curve of deaths" and the associated "curve of lives."

13. Employing the distances (p_1, p_2, p_3) from the sides of an equilateral triangle as the trilinear coördinates of an arbitrary point in the plane of the triangle, one can solve several boundary value problems involving the Laplacian operator. Professor Holl derived known results in polynomial form for the torsion and flexure problems of prisms whose cross sections are the equilateral triangle and circle, the circular annulus and semi-circle. A slight generalization was obtained by generalizing the reference triangle to be an isosceles triangle.

14. The purpose of Professor Oberg's paper was to exhibit a simple rule for solving integral equations of the type

$$\phi(x) = f(x) - \lambda \int_{-1}^{+1} K(x \pm t)\phi(t)dt$$

when λ is not a characteristic number, and the functions $f(x)$ and $K(x \pm t)$ are assumed to be continuous in the intervals $(-1 \leqq x \leqq 1)$ and $(-1 \leqq x,\ t \leqq 1)$, respectively. Previous writers have solved the same equation by reducing it to a differential equation or by the application of the general Fredholm method. In the present instance, the solution was found by means of least squares.

15. In this paper Professor Allen considered probabilities which depend on the outcome or on the probability of previous events. Certain classical problems

are of this type, but the Markoff chain was the first to be intensively studied. Onicescu and Mihoc have considered a much more general case, investigating particularly the existence of limiting probabilities. The present paper was concerned especially with Pólya's "contagion in probability," and Eggenberger's and Linder's developments of it. Tests applicable to empirical data, to determine the existence and magnitude of hysteretic influence, were discussed.

16. Little progress was made in unifying various mathematical domains until the dawn of the twentieth century. Professor Graber spoke of the work of E. H. Moore, of the Logistic Formalists exemplified by Russell, Whitehead, and Keyser, of the Intuitionist School represented by Brouwer, Weyl, and Bell. For applied mathematics or meta-mathematics all we need to ask is (with Kempe) whether, if given a certain class of objects and a certain class of relations, the ordered groups of these objects do or do not satisfy the relations. The acceptance of this definition permits us to employ deduction, induction, and experimentation in building up our mathematical system.

17. Professor Smith showed that under certain restrictions the small roots of the confluent hypergeometric functions, and the associated Whittaker functions $W_{k,m}$ could be given by a formula obtained from an expansion of the functions in powers of the parameter $1/4k^2$ which satisfies the differential equation which defines the functions.

CORNELIUS GOUWENS, *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The fifteenth annual meeting of the Nebraska Section of the Mathematical Association of America was held in Hastings, Nebraska, on Friday afternoon, May 6, 1938, in conjunction with the annual meeting of the Nebraska Academy of Sciences. Professor R. M. McDill of Hastings College was chairman.

The attendance was thirty-four, including the following fifteen members of the Association: A. K. Bettinger, W. C. Brenke, A. L. Candy, W. A. Dwyer, J. M. Earl, M. G. Gaba, Emma E. Hanthorn, W. G. Leavitt, F. E. Marrin, R. M. McDill, U. G. Mitchell, J. D. Novak, T. A. Pierce, Lulu L. Runge, S. T. Sanders, Jr.

Officers for the ensuing year were elected as follows: Chairman, W. C. Brenke, University of Nebraska; Secretary-Treasurer, T. A. Pierce, University of Nebraska; Member of Executive Committee, J. M. Earl, Municipal University of Omaha.

The following program was presented:

1. "How may mathematics secure and hold in the curriculum the position which it deserves?" by Professor U. G. Mitchell, University of Kansas.

2. "A note on the problem of interpolation by polynomials" by Professor J. M. Earl, Municipal University of Omaha.

3. "Certain facts about trigonometric series" by Professor W. C. Brenke, University of Nebraska.

4. "A simple approximation for π" by Professor M. G. Gaba, University of Nebraska.

5. "Planetary orbits in relativity" by W. G. Leavitt, University of Nebraska.

6. "Orthogonal and Appell polynomials" by Dr. M. S. Webster, University of Nebraska, introduced by the Secretary.

7. "The mathematics of factor analysis" by J. D. Novak, University of Nebraska.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Mitchell traced briefly the changes in the position of mathematics in secondary education during the last fifty years, pointing out the difficulties encountered in the attempt to give all children the same kind of schooling. The present situation is one important phase of the problem of mass education. The tremendous increase in high school enrollment since 1900 has forced a reorganization of the mathematical curriculum and further differentiation of both courses of study and methods of teaching must be made to meet the needs of individual and group differences. The development of an adequate evaluation program, which shall determine scientifically and objectively the effects of schooling upon the entire personality of pupils, is the greatest need and such a program is already under way.

2. Professor Earl dealt with interpolation to a given function of x by means of polynomials of degree $n(n=0, 1, 2, 3, \cdots)$ on the interval $(-1, 1)$. These polynomials assume the value of the function at $(n+1)$ equally spaced points. For a sub-interval whose length is of the order of the negative one half power of n, an upper bound for the magnitude of the polynomial of degree n is obtained; this upper bound contains n and an upper bound for the magnitude of the given function.

3. In this paper Professor Brenke considered the expansion of some specific functions in trigonometric series summable $(C, 1)$.

4. This paper appeared in the June-July issue of this MONTHLY.

5. Mr. Leavitt gave a brief description of the general relativity theory, leading through a specialization of the general line element to the differential equation of the planetary orbit. An approximation yielded the observed advance in perihelion. The integration of the orbital equation in terms of elliptic functions was considered and the shape of the orbit under various initial conditions was described.

6. Systems of Appell polynomials and systems of orthogonal polynomials are well known. Dr. Webster gave a concise proof, using recurrence relations, that there is essentially only one system of Appell polynomials which are also orthogonal.

7. Recent investigations in psychological measurements have made use of an increasing amount of higher mathematics. In particular, Professor L. L. Thurstone of the University of Chicago has developed a theory of multiple

factor analysis in which matrices and the geometry of hyperspaces play a fundamental part. Mr. Novak presented an outline of factor analysis and discussed some of the mathematics encountered in this theory.

T. A. PIERCE, *Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on Saturday, May 7, 1938. The chairman, Professor Gillie A. Larew, presided over both sessions, morning and afternoon. The Section was happy to have Professor Hans Rademacher of the University of Pennsylvania as guest-speaker.

The attendance was fifty-four, including the following thirty-one members of the Association: O. S. Adams, N. H. Ball, G. A. Bingley, Archie Blake, C. C. Bramble, Paul Capron, Randolph Church, G. R. Clements, Abraham Cohen, A. E. Currier, L. S. Dederick, J. A. Duerksen, P. J. Federico, E. J. Finan, Michael Goldberg, L. M. Kells, W. D. Lambert, A. E. Landry, Gillie A. Larew, C. L. Leiper, C. M. Lennahan, S. B. Littauer, T. W. Moore, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, J. B. Scarborough, F. W. Sohon, John Tyler, John Williamson.

At a business meeting the following officers were elected for next year: Chairman, Michael Goldberg, Bureau of Ordnance, Navy Department; Secretary-Treasurer, S. B. Littauer, U. S. Naval Academy; additional members of the Executive Committee, Mildred E. Taylor, Mary Baldwin College, and Florence P. Lewis, Goucher College. An invitation to hold the next meeting at the University of Maryland, College Park, Maryland, on December 10, 1938, was accepted.

After an address of welcome by Admiral Wilson Brown, Superintendent of the Naval Academy, the following five papers were read:

1. "The relation of the range of a sample to the standard deviation of the population" by Dr. L. S. Dederick, Aberdeen Proving Ground.

2. "Some naval tactics in vector analysis" by Professor C. H. Rawlins, Jr., Postgraduate School, U. S. Naval Academy.

3. "Infinite continued radicals and iteration of polynomials" by Aaron Herschfeld, Social Security Board, Washington, D. C., introduced by the Secretary.

4. "Mathematical applications common to practical meteorology" by Lieut. H. B. Hutchinson, Postgraduate School, U. S. Naval Academy, introduced by Professor Root.

5. "Partitio numerorum" by Professor Hans Rademacher, University of Pennsylvania, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Dederick gave a simple expression in terms of a definite integral for the probability that the range of a sample does not exceed any given multiple of the standard deviation of the population from which it is drawn.

2. Professor Rawlins applied methods of vector algebra and calculus to a few problems in tactical graphics in which certain aircraft maneuvers, relative to a given set of moving ships, were to be performed in minimum length of time. From the resulting vector equations there were deduced suitable geometrical constructions for solving the problems.

3. Mr. Herschfeld showed that the infinite set of possible representations of a number by an infinite continued radical employing only positive integers has the power of the continuum. He used some of these representations to generalize the results of Pincherle and Lucas on quadratic polynomials to obtain explicit expressions for the iterates of special polynomials of any degree.

4. Meteorology originally attracted the keenest scientific minds. It failed to hold them during the nineteenth century. Investigation ceased, practical weather forecasting seemed to be the primary objective. Recently the science has been rejuvenated through the availability of exhaustive data from the third dimension of the atmosphere. These data are handled in the works of V. and J. Bjerknes, C. G. Rossby, S. Petterssen, and others by use of the methods of higher mathematics. Some of these applications were discussed by Lieutenant Hutchinson who showed the substantiation of the theory by the observational facts and the extent of their practical value to the field meteorologist of to-day.

5. Professor Rademacher gave a historical account of the problems involved in the partition of numbers. These problems date back to Euler who, in his "Introductio in analysin infinitorum" (1748), included a chapter "De partitione numerorum." In recent times they have been classified as forming a part of the additive theory of numbers. Euler's methods and results were sketched, in particular his "pentagonal number theorem," in which appears, for the first time, a special case of a theta-function. The speaker then referred to the modern analytic methods introduced into this field by a paper of G. H. Hardy and S. Ramanujan (1917), which contains the asymptotic formula for $p(n)$, the number of unrestricted partitions of n. He finally mentioned the convergent series for $p(n)$ published by himself in 1937, and recent results by Dr. H. S. Zuckerman and himself concerning identities which Ramanujan had conjectured.

MICHAEL GOLDBERG, *Secretary*

THE HISTORY OF BLISSARD'S SYMBOLIC METHOD, WITH A SKETCH OF ITS INVENTOR'S LIFE*

E. T. BELL, California Institute of Technology

1. It sometimes happens in the history of mathematics that the credit for a particular method is commonly ascribed to another than its originator. In the interests of historical justice, such oversights should be corrected, where the facts are known. A conspicuous instance is the extremely useful symbolic method expounded and widely applied by Edouard Lucas [**3, 5**],† and usually attributed to him. The kernel of this symbolic method is described below, §2.

This representative notation [**1**], to use the designation of its inventor, or symbolic method, as Lucas [**3, 5**] called it, or umbral calculus [**8,** vol. 31], was fully developed in 1861 by John Blissard [**1**], in a mathematical journal with a wide circulation among mathematicians, fifteen years before Lucas published his first papers [**3, 4**] on the subject. Although J. W. L. Glaisher [**6,** p. 159] called attention to Blissard's indisputable priority forty years ago, it seems still to be not generally known to those who use or refer to the symbolic method. Thus, for example, the symbolic treatment of the Bernoulli numbers, first given by Blissard [**1,** vol. 4], is assigned to Lucas [**3**]. Regarding this treatment, L. E. Dickson [**7**] expressed the opinion of most students of the Bernoulli (and allied) numbers when he stated that, "*Bernouillian numbers* are most conveniently employed in the symbolic notation of Lucas. . . ."

There are also occasional misconceptions of the abstract nature of Blissard's method itself. It is sometimes thought to be non-elementary, or transcendental, because it may, but need not, make use of strictly formal operations on power series. Such a use of power series is algebraic, not analytic, as no limiting process is involved, and convergence is irrelevant. As pointed out elsewhere [**8**], the appropriate algebra of power series, and hence also Blissard's tacit use of this algebra, is no more 'transcendental' than mathematical induction is. In fact, both the 'elementary' method of N. Nielsen [**10**] for the Bernoulli numbers, and Blissard's symbolic method, contain precisely the same amount of transcendence: each must use mathematical induction (as must any algebraic algorithm on countable sequences of elements) for a complete proof of the results obtained by it, and neither requires any other appeal to the infinite. Hence Nielsen's claim [**10,** p. IX] that his "elementary theory of the Bernoulli Numbers . . . is much more fundamental than the symbolic method, developed notably by Lucas . . . ," is without foundation. Likewise for his justification of this claim, "because a symmetric polynomial [as defined by him] gives immediately results concerning all the numbers B_n, E_n, T_n." Blissard's method gives at once [**9**] the 15 possible most general formulas connecting any 1, 2, 3, or 4 of the sequences of Bernoulli, Euler, Genocchi, and Lucas; and as these formulas, in the cases where precisely one of the four sequences is concerned, are both necessary and

* Presented for Slaught Memorial Volume of the MONTHLY.

† Numbers in square brackets refer to the list of references at the end of the paper.

sufficient to define each sequence, it follows that the definition by symmetric polynomials contains nothing that is not implicit in the completely general formulas of Blissard's type. Moreover, the symbolic treatment of the Bernoulli and allied numbers is but one detail in the applications of the symbolic method.

2. For a statement of the essentials of Blissard's method, any of his papers [1], or Lucas [5], or Bell [11], may be consulted. A recapitulation of the fundamental definition and its immediate consequences will suffice here to recall the nature of Blissard's calculus.

Let $a_n, b_n, \cdots, x_n, \cdots$ $(n=0, 1, \cdots)$ be any sequences of real or complex numbers; such numbers will always be denoted by small Latin letters *with suffixes*. A letter, say x, *without a suffix*, is the *representative*, or *umbra*, of the sequence x_n $(n=0, 1, \cdots)$ derived (notationally) from x in an obvious manner. Two umbrae, x, y, are defined to be *equal*, $x=y$, if and only if $x_n=y_n$ $(n=0, 1, \cdots)$. It follows that umbral equality has the properties of equality in abstract algebra (rings, fields, etc.): for any x, y one of $x=y$, $x\neq y$ is true; umbral equality is symmetric, reflexive, and transitive.

Blissard [1] observed that if x_n is written as a symbolic, or umbral, power, x^n, such powers have some of the properties of powers as in abstract algebra. In particular, if $x=y$, then $x^n=y^n$, $n=0, 1, \cdots$; that is, $x_n=y_n$, $n=0, 1, \cdots$. This is merely half the definition of equality restated; nevertheless, it is the germ of the entire calculus. For if x, y, z are any umbrae such that $x\neq y$ and

$$z_n = \sum_{s=0}^{n} (n, s) x_s y_{n-s}, \qquad n = 0, 1, \cdots,$$

where (n, s) is the binomial coefficient $n!/s!(n-s)!$, $(0, 0)=1$, we may write

$$z = x + y,$$

understanding by this,

$$z_n \equiv z^n = (x + y)^n, \qquad n = 0, 1, \cdots,$$

in which the umbral binomial is to be expanded as if x, y were ordinaries (real or complex numbers), and, after expansion, all exponents are to be degraded to suffixes. It is emphasized that in all rational operations on umbrae, as in the expansion of $(x+y)^n$, the exponents 0, 1 must be treated precisely as any other non-negative integer exponents, $x^0\equiv x_0$, $x^1\equiv x_1$, $(x+y)^0\equiv x_0y_0$, $(x+y)^1\equiv x_0y_1 + x_1y_0$.

The umbral binomial theorem just described can be extended in an obvious way to multinomials. Thus if $x, y, \cdots, u$ are all unequal,

$$z_n = \sum M_{\alpha,\beta,\cdots\gamma} x_\alpha y_\beta \cdots u_\gamma,$$

where the M's are the multinomial coefficients occurring in the expansion of $(X+Y+\cdots+U)^n$, is equivalent to the umbral equality

$$z = x + y + \cdots + u,$$

that is, to

$$z_n \equiv z^n = (x + y + \cdots + u)^n, \qquad n = 0, 1, \cdots.$$

Hence, if for any umbra t, and a real complex variable θ, $\exp t\theta$ denotes the formal MacLaurin series on the right of the next,

$$\exp t\theta \equiv \sum_{n=0}^{\infty} t^n\theta^n/n! \equiv \sum_{n=0}^{\infty} t_n\theta^n/n!,$$

it follows that the three statements

$$\exp z\theta = \exp x\theta \exp y\theta \cdots \exp u\theta,$$
$$\exp z\theta = \exp (x + y + \cdots + u)\theta,$$
$$z = x + y + \cdots + u,$$

are equivalent, when, as in the algebra of formal power series [**8**, vol. 25], equality of such series, $\exp t\theta = \exp v\theta$, is defined to be equivalent to $t = v$.

From here on the umbral calculus is developed in easy detail; it includes an umbral differential and integral calculus as useful implements. It is not necessary here to explain the simple modifications required when two or more of $x, y, \cdots, u$ above are equal. There are immediate (and at present trivial) extensions of the method to sequences of elements from any commutative ring.

3. Lucas [**5**, p. 205] introduces his account of the symbolic calculus with some historical remarks which are immaterial for the calculus, and hence irrelevant and possibly misleading. He says (translation), "This method is already old; it is found as a mnemonic procedure in the writings of Leibniz, *for the successive derivatives of a product of two or of several factors*; it is found again in Taylor's series extended to the case of several variables. · · · Developed subsequently by Laplace, by Vandermonde, and by Herschel, it has been considerably augmented by the works of Cayley and of Sylvester, in the theory of forms.

"In a work entitled [*A Treatise on the*] *Calculus of Operations*, [1855, Robert] Carmichael has expounded the general methods of this rapid calculus; his developments refer above all to the use of symbols of operation, such as those of the sum and difference calculus, Σ and Δ, and those of the differential and integral calculus. But the symbolic method, which we employ here, *differs from the preceding in this respect: the symbols which we consider denote quantities* [real or complex numbers] *and not operations* . . . . The application of this method has enabled us to simplify, in a very notable manner, the reasoning and the results in the calculus of sums and differences, in the numbers of Bernoulli and Euler, in the theory of recurring series and, consequently, in the general theory of functions. By *our* method, the developments assume a more concise and more condensed form, which leads to successive and endless generalizations of the properties which concern numbers, and of the formulas which contain them."

In the preceding extract all italics except the title of Carmichael's book (which contains nothing relating to Blissard's method) are the translator's. The second italicized passage cancels the first, so far as Blissard's method is concerned, and makes all the historical citations irrelevant. The separation of symbols, as some older writers called the symbolic method of Leibniz and his successors, to which Lucas is referring in his historical remarks, *differs* from what Lucas calls 'our' method, as he himself takes pains to point out. A check of the sources mentioned by Lucas, also the like for the history of Leibniz' method from the historical references given by H. T. Davis [**12**], shows conclusively that none of the sources has anything whatever to do with Blissard's symbolic method. None of the authors mentioned has an example of the method. The italicized 'our' should be replaced by 'Blissard's.' It is not clear why Lucas gave all this irrelevant history; the matters to which it refers are not considered in his chapter on the symbolic method.

However, it has recently been shown by L. I. Neikirk [**13**] that the method of Leibniz, Cayley's hyperdeterminant notation, Aronhold's symbolic method in algebraic invariants and covariants, and Blissard's calculus, have a common *mathematical* (as distinguished from historical) source in symbolic differentiation. But here again it must be emphasized that Blissard's advance consisted in taking the final step of expunging all symbols of operation except addition subtraction, multiplication, and raising to positive integral powers from the formulas of his calculus and, as noted by Lucas in his account quoted above, using in addition only symbols which denote numbers themselves (independently of their possible origin as coefficients obtainable operationally from power series). If the common source referred to was known before 1937, there is no trace of it in the published literature. It seems curious that Lucas, who was widely read in the historical literature of the theory of numbers, and who did much notable research in that literature, and who further contributed papers written in English to English mathematical periodicals, should have overlooked Blissard's papers in one of the most widely circulated of such periodicals. Sylvester, however, is said to have surpassed even this obliviousness to what was going on or had gone on in his own field: he refused to believe that he had ever made some of his own discoveries.

Lucas also (*loc. cit.*) advocates passage from the symbolic notation to the ordinary when "the chain of the reasoning leaves a certain obscurity in the mind." There is no obscurity when the simple bases of the method are explicitly stated, which Lucas neglected to do. The root of the method is umbral equality, as defined in §2. The current practice of explicitly stating the definitions and postulates for a given method or theory was not customary when Lucas wrote. Attention to this usual precaution of clarity and definiteness removes obscurities, especially those that enter with unstated definitions and ignored postulates which must be inferred from ambiguous contexts.

Before proceeding to the sketch of Blissard's life, it will be of interest to recall some remarks of Glaisher in his paper [**6**, p. 159] on the Bernoullian function.

"The results in the nine preceding sections [of the paper in question], in so far as they relate to the Bernoullian numbers, . . . , are due to the late Mr. John Blissard, who published them in 1861 in [**1**]. Fifteen years later they were rediscovered by the late M. Eduard [*sic*] Lucas, and published by him in [**3**] and elsewhere. . . .

"In subsequent papers in the *Quarterly Journal*, Blissard developed in a systematic manner his symbolic method of investigation under the name Representative Notation.

"Lucas's papers attracted considerable attention, and the symbolic treatment of formulae relating to the Bernoullian and Eulerian numbers formed the subject of various papers and notes by M. Césaro [*sic*] and others. It does not appear, however, to have been pointed out that he had been anticipated by Blissard."

Cesàro (from remarks in his papers) credited Lucas with the method. Blissard was fully aware that he had devised a powerful algorithm, and claimed it for himself [**1**, vol. 5, p. 75], as he had the historical right to do. He says: "The subject of generic equation[s] appears to furnish a mine of analytical research, in which, by aid of my notation, as a fitting tool to work with, whatever direction may be taken, new and highly general results appear capable of being turned up in considerable abundance." 'Generic equations' are now called symbolic or umbral equations; thus, a generic equation for the Bernoulli numbers is Blissard's $(B+1)^n-B^n=0$, $n=0, 1, \cdots$, or $B+1=B$.

4. Mathematically, Blissard was 'a man of one book.' He did in fact project a book on his invention, to unite his papers and give further applications of his method; but the constant demands on his time as a clergyman in an English village of the mid-nineteenth century postponed the book indefinitely, and it was never even begun. His life takes us back to an era that now seems almost as remote as the seventeenth century.*

The name is pronounced Blissard′, and has been spelt variously Bloissart, Blizard, Blezard since the Black Prince in 1356, after the battle of Poitiers, gave his knights, one of whom was a Blissard, their coats of arms. For about 300 years the eldest son of the eldest son was named John and was a minister of the Church of England; the mathematician was almost in this tradition. There was also a medical tradition in the family; a biography of the distinguished English surgeon of the eighteenth and early nineteenth centuries, Sir William Blizard, will be found in the Imperial Dictionary of National Biography. This Blizard was active in the founding of the Royal College of Surgeons of London, where he was professor of anatomy. Mathematical talent appeared in two generations; both the subject of this sketch and his son specialized in mathematics at Cambridge, and both for a time did mathematical coaching. The son also went in

* The genealogical data in the following condensed account were supplied by John Morton Blissard, M.D., and his wife, of La Jara, Colorado. The few details of Blissard's life given below are summarized from a letter to the writer from Mrs. Evelyn Blissard Hooper, of Berkshire, England, a grand-daughter of the mathematician Blissard. This correspondent had access to family records and letters.

strongly for chess, and problems attributed to him are still to be found in chess manuals.

The mathematician John Blissard, 1803–1875, was born in Northampton, England, where his father was a physician "in good practise" [**2**]. He died at Hampstead Norreys, or Norris (from which inconspicuous village his mathematical writings are dated), after having served the local church, first as curate, then as vicar, for forty-six years. It may seem strange to some today that an inventive mathematician could have been content to live out his days as the pastor in a drowsy village when good (perhaps too good) Queen Victoria was at her very best, with all that superlative excellence meant for rustic society in England. But such was the fact, and Blissard was never heard to express any discontent with his lot, although "he was considered buried in that little village by those who saw his powers."

Blissard's Cambridge career was only moderately distinguished. He went into residence at St. John's College in 1822, and in the same year became prizeman and Mount Stephen exhibitioner. In the mathematical tripos of 1826 he came out senior optime, and was graduated B.A. His rank in the tripos, however, was not a just measure of his mathematical ability, as he had already done what would now be called independent research as an undergraduate. In 1827, he married Martha Morton, of the same family as the Scotch Earl of Morton. To anticipate, there were twelve children.

Having been ordained in 1827, Blissard served two years as a curate in a Bedfordshire parish, moving on "to be shelved" in Hampstead Norreys where, for a spell of fourteen years, he was curate. On the death of the vicar, Blissard was promoted—if that is the right word—to vicar in 1843. "Mr. Blissard was so beloved by his parishioners, that they presented a petition in his favor to the Marquis of Downshire (then patron of the living), who graciously presented it to Mr. Blissard" [**2**]. We are back with George Eliot in an age that is farther from us than the pyramids of Egypt. According to the same source, "Mr. Blissard was a clever mathematician, and wrote some erudite works on the subject." This slightly fatuous comment is what might be expected from our own Middletowns today; so possibly the Victorian past of rural England is less remote than it seems at first glance.

Blissard kept his soul alive—primarily by his ministrations to the parish, of course—by coaching private pupils in mathematics for Cambridge examinations, in which some of them, including two sons of one of Queen Victoria's personal physicians, did remarkably well. He was fifty-eight when his symbolic method was published. His daughter has recorded how this late productivity under adverse conditions may have been possible. In addition to his professional preoccupation with "things of the spirit, another part of his mind dwelt deeply in things mathematical, and he always had some abstruse problems or 'inventions' in his thoughts which one day he would bring out in a book."

Naturally, such a schism in fundamental loyalties resulted in frequent absentmindedness, especially as Blissard aged. One anecdote of many may serve to give a picture of this thoroughly amiable Victorian clergyman who was

mathematician enough to decline fees from his promising pupils when they happened to be hard up, and who kept mathematically alive long after an age at which men in easier positions often lose their intellectual interests.

In Blissard's day, churchgoers thought they were treated shabbily if the sermons were less than epic length, and Blissard agreed with them, once unwittingly. One Sunday he took as his text the whole of the twenty-third psalm, an adequate yardstick for what was to follow. Each sentence of the entire psalm provided a sub-text for a sub-sermon of moderate length, which the conscientious minister duly preached. By the time he reached "He restoreth my soul," he was just getting thoroughly warmed up. The phrase furnished the text for a complete address. By the end of this address, the preacher had forgotten its text. For the next lap of his marathon of a sermon he again gave out, "He restoreth my soul," and delivered another address, completely different from the first, on the same topic. Again he lost his cue, gave out the same sub-text a third time, and "preached a compact and beautiful little sermon on the same words without impinging upon anything he had said before. All three addresses," according to his wife, who, by the end of the third lap had caught the preacher's mildly excited eye, "were quite different and all of them quite perfect each in its own way. (And the people didn't mind a bit!)." There were giants in those days, and not all of them filled pulpits.

In a way that our generation cannot recover, Blissard's life was idyllic. But even in his tranquil rural paradise, the old Adam—or the ever-young Eve—leers at us through the lush vegetation. With an unconscious irony that speaks volumes of repressed social conflict, the official obituary [**2**] records that, "Throughout his life he [Blissard] scarcely made an enemy, and although his house was once set on fire by an evil-minded servant he forgave her, when she admitted her guilt."

In its own time and in its own degree, the laborious life of this inventive man illustrates a recent observation of the late J. N. W. Sullivan [**14**, p. 271]: "It seems very odd to reflect that these Bernoullis and Gauss might never have been known as mathematicians at all. There are plenty of instances of a genuinely promising young mathematician becoming by what seems mere accident of circumstance a lawyer or a churchman. The spectacle of square pegs in round holes is not so very uncommon. . . . "

It would be socially interesting to know how many potential mathematicians and other possibly useful citizens are stuck at this moment in unsuitable jobs in uncongenial holes in the United States. With this information in hand, it might be possible to dissuade certain "evil-minded" servants of society from their intention to burn down the schoolhouse.

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ON THE LINE OF IMAGES*

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Introduction. If we take any point T on the circumcircle of a triangle $A_1A_2A_3$ and reflect this point in the sides of the triangle we obtain three points lying on a line—the line of images of the point T. With the aid of this line we shall show a relationship which exists among certain known theorems and state some consequences which are believed to be entirely new. The method is analytic. In the study of the triangle it is convenient to take its circumcircle as the base circle and to let the coördinates of the vertices A_i $(i=1, 2, 3)$ be turns t_i, *i.e.*, complex numbers whose moduli are unity. We consider the t_i as roots of the equation $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$, *i.e.*,

$$\sigma_1 = t_1 + t_2 + t_3, \qquad \sigma_2 = t_1t_2 + t_2t_3 + t_3t_1, \qquad \sigma_3 = t_1t_2t_3.$$

The conjugates of these elementary symmetric functions of the t_i are

$$\bar{\sigma}_1 = \frac{\sigma_2}{\sigma_3}, \qquad \bar{\sigma}_2 = \frac{\sigma_1}{\sigma_3}, \qquad \bar{\sigma}_3 = \frac{1}{\sigma_3}\cdot$$

With this notation the parametric equation of the nine-point circle, with center at $\sigma_1/2$, and radius half of the base circle, is

$$x = \frac{(\sigma_1 - t)}{2}\cdot$$

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* Presented at the meeting of the Ohio Section of the Association at Columbus, March 31, 1938.

1. The line of images. The equation of any line may be written in the self-conjugate form

$$\frac{x}{b} + \frac{\bar{x}}{\bar{b}} = 1. \tag{1.1}$$

The perpendicular to this line through a point P (with coördinate p) is

$$\frac{x}{b} - \frac{\bar{x}}{\bar{b}} = \frac{p}{b} - \frac{\bar{p}}{\bar{b}}.$$

The intersection Q of these two lines is

$$x = \frac{b}{2}\left(1 + \frac{p}{b} - \frac{\bar{p}}{\bar{b}}\right).$$

Now the image x_1 of P in the line (1.1) is such that Q is the midpoint of x_1P, whence

$$x_1 = b\left(1 - \frac{\bar{p}}{\bar{b}}\right).$$

We note that this could be written down immediately from (1.1) by substituting $\bar{p}$ for $\bar{x}$ and solving for x.

The equation of the side A_2A_3 of the triangle $A_1A_2A_3$ is

$$x + t_2t_3\bar{x} = t_2 + t_3,$$

whence, since $\bar{T} = 1/T$, the image of the point T (with coördinate T) is

$$x = t_2 + t_3 - \frac{t_2t_3}{T}$$

$$= \sigma_1 - t_1 - \frac{\sigma_3}{Tt_1}.$$

The three images of T in the sides of the triangle $A_1A_2A_3$ lie on the line, with parameter t,

$$x = \sigma_1 - t - \frac{\sigma_3}{Tt}.$$

Since

$$\bar{x} = \frac{\sigma_2}{\sigma_3} - \frac{1}{t} - \frac{Tt}{\sigma_3},$$

upon elimination of the parameter t between these equations we obtain as the self-conjugate equation of the line of images of the point T,

$$Tx - \sigma_3\bar{x} = T\sigma_1 - \sigma_2. \tag{1.2}$$

Since this equation is satisfied for $x=\sigma_1$, we see that *the line of images of any point on the circumcircle of $A_1A_2A_3$ passes through the orthocenter of $A_1A_2A_3$* [1].* We note also that the lines of images of any two diametrically opposite points on the circumcircle are perpendicular and that when $T=t_i$, the line of images is the altitude of the triangle. More generally the line of images of T is parallel to the Simson line of T and hence the angle between the lines of images of two points equals the angle inscribed in the arc of the circumcircle between the points.

Given any line on the orthocenter of $A_1A_2A_3$ to find the point T for which the given line is its line of images we proceed as follows: Draw through any vertex A_i a line parallel to the line of images intersecting the circumcircle at a point V. The perpendicular from V to the side A_jA_k of the triangle will intersect the circumcircle at the required point T.

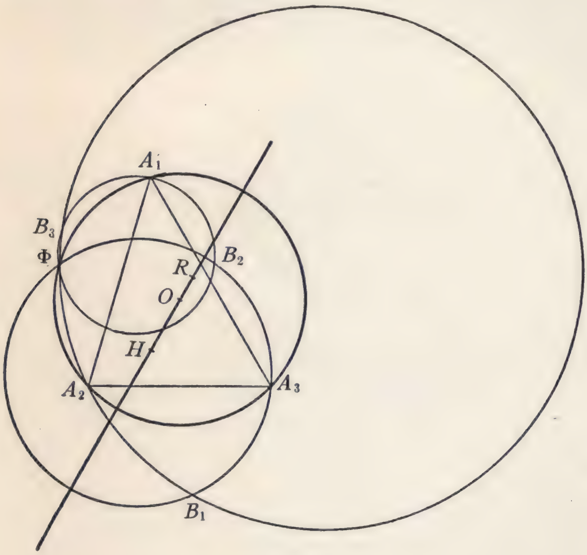

FIG. 1

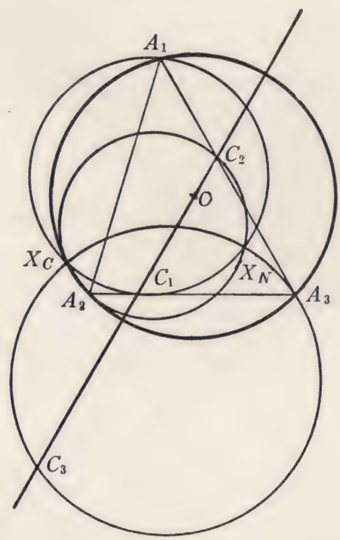

FIG. 2

Since the equation of the Euler line is

$$\sigma_2 x - \sigma_1\sigma_3\bar{x} = 0$$

we see that the point whose line of images is the Euler line has for its coördinate σ_2/σ_1. But this is the coördinate of the point of Feuerbach for the tangential triangle† of $A_1A_2A_3$ [2]. We shall designate this point by the symbol Φ.

2. A property of the line of images. We shall generalize a theorem of Canon. Given the images B_1, B_2, B_3 of the circumcenter O of the triangle $A_1A_2A_3$ in

* Numbers in brackets refer to references at the end of the paper.

† This is the point of tangency of the inscribed and nine-point circles for the triangle formed by the tangents to the circumcircle at the points A_1, A_2, and A_3.

the sides A_2A_3, A_3A_1, A_1A_2; show that the circles $B_1B_2A_3$, $B_2B_3A_1$, $B_3B_1A_2$ meet at the same point on the circumcircles [3]. Let P be any point in the plane with coördinate p. Its image B_1 in the side A_2A_3 has as coördinate, $t_2+t_3-t_2t_3\bar{p}$.

Consider the equation

$$(t_2 + t_3)x = \sigma_2 - \sigma_3\bar{p} + (t_2t_3 - \sigma_3\bar{p})t. \tag{2.1}$$

For $t=-1$, $t_3t_2^{-1}$, and $t_2t_3^{-1}$ respectively, this circle passes through the points A_1, B_2, and B_3. Hence this is the equation of the circumcircle of $B_2B_3A_1$. Also for the turn value

$$t = \frac{(\sigma_2 - \sigma_3\bar{p})(p - t_1)}{(\sigma_1 - p)(t_2t_3 - \sigma_3\bar{p})}, \qquad x = \frac{\sigma_2 - \sigma_3\bar{p}}{\sigma_1 - p},$$

a point M on the circumcircle of $A_1A_2A_3$. Since the coördinate of this point is symmetrical in t_i, the circumcircles $B_1B_2A_3$ and $B_3B_1A_2$ are likewise on M. Further, if R be any point on the line HP, where H is the orthocenter of $A_1A_2A_3$, and we take for coördinate of R,

$$r = \frac{H + \lambda P}{1 + \lambda} = \frac{\sigma_1 + \lambda p}{1 + \lambda},$$

the coördinate of M becomes $(\sigma_2-\sigma_3\bar{r})/(\sigma_1-r)$. Substituting

$$r = \frac{\sigma_1 + \lambda p}{1 + \lambda} \qquad \text{and} \qquad \bar{r} = \frac{\sigma_2 + \lambda\sigma_3\bar{p}}{\sigma_3(1 + \lambda)}$$

in this expression for M we obtain $(\sigma_2-\sigma_3\bar{p})/(\sigma_1-p)$. Since this is independent of the parameter λ we can state this generalization of Canon's theorem: *For every point R on the line HP, where H is the orthocenter of $A_1A_2A_3$ and P is any point in the plane, if we determine the images B_1, B_2, B_3 of R in the sides A_2A_3, A_3A_1, A_1A_2, the four circles $B_1B_2A_3$, $B_2B_3A_1$, $B_3B_1A_2$, and $A_1A_2A_3$ meet a fixed point M.* The circles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$, and $B_1B_2B_3$ likewise meet at a point. A simple proof of this latter statement has been given by the author [4].

The equation of the line HP is

$$(\sigma_2 - \sigma_3\bar{p})x - \sigma_3(\sigma_1 - p)\bar{x} = \sigma_2 p - \sigma_1\sigma_3\bar{p}. \tag{2.2}$$

Comparing this with (1.2) we see that HP is the *line of images of the point M*. Hence given HP, we have a simple method for constructing M and vice-versa. But we have proved the following: *For every point R on the line of images of a point T, if we determine the images B_1, B_2, B_3 of R in the sides of the triangle $A_1A_2A_3$, the circles $B_1B_2A_3$, $B_2B_3A_1$, and $B_3B_1A_2$ intersect at T.* This theorem is illustrated in Figure 1 for the point Φ.

3. A theorem of Blanc. If any transversal be drawn through O, the circumcenter of $A_1A_2A_3$, cutting the sides A_2A_3, A_3A_1, A_1A_2 in the points C_1, C_2, and C_3 respectively, the three circles with A_iC_i as diameters meet in two points, one

on the circumcircle and the other on the nine-point circle of $A_1A_2A_3$. Their common chord passes through H, the orthocenter of $A_1A_2A_3$ [**5**].

If we join the point T on the circumcircle to O, the equation of the transversal OT is

$$x - T^2\bar{x} = 0. \tag{3.1}$$

This cuts the side A_2A_3 in the point C_1, with coördinate

$$c_1 = \frac{T^2(t_2 + t_3)}{t_2t_3 + T^2}.$$

The equation of the circle on A_1C_1 as diameter is

$$2(t_2t_3 + T^2)x = T^2\sigma_1 + \sigma_3 + (T^2\sigma_1 - 2T^2t_1 - \sigma_3)t.$$

The three circles with A_iC_i as diameters meet at the two points

$$X_C = \frac{\sigma_3 + \sigma_1T^2}{\sigma_2 + T^2} \quad \text{and} \quad X_N = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right). \tag{3.2}$$

The first point is on the circumcircle of $A_1A_2A_3$, while the second point is on the nine-point circle (Figure 2). The equation of the common chord of the three circles is

$$T^2(\sigma_2 + T^2)x - \sigma_3(\sigma_3 + \sigma_1T^2)\bar{x} = \sigma_1T^4 - \sigma_2\sigma_3. \tag{3.3}$$

Since this equation is satisfied for $x=\sigma_1$, the chord (3.3) passes through the orthocenter of $A_1A_2A_3$. Now the image of the point X_C in the given transversal (3.1) is

$$\alpha = \frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1T^2}. \tag{3.4}$$

Comparing this with (3.3) we see that *the common chord of the three circles is the line of images of the point α, the image of X_C in the given transversal OT*. Consequently, it is possible, given any line on H, to construct the transversal through O which will cut the sides of $A_1A_2A_3$ in points C_i such that the common chord of the circles on A_iC_i will be the given line. We proceed as follows: Find the point α whose line of images is the given line. This given line intersects the circumcircle at two points R_1 and R_2. Now the perpendicular bisector of αR_i is the required transversal, if the common intersection of the three circles is to be R_i.

We state without proof the following properties of the points α and X_C. Let A_iV_i be parallels drawn through the vertices A_i to the transversal OT, cutting the circumcircle at the points V_i; then the three lines C_iV_i meet at α. If A_i' be the points on the circumcircle diametrically opposite to A_i, the lines $A_i'C_i$ meet at X_C.

4. The isogonal conjugate of OT. It is well known that the isogonal conjugate of any diameter of the circumcircle $A_1A_2A_3$ is an equilateral hyperbola cir-

cumscribing the triangle $A_1A_2A_3$ [6]. In our notation any two isogonal conjugate points x and y are connected by the relation [1, p. 196]

$$x + y + \sigma_3\bar{x}\bar{y} = \sigma_1$$

which, solved for y in terms of x, gives

$$y = \frac{\sigma_1 - x - \sigma_2\bar{x} + \sigma_3\bar{x}^2}{1 - x\bar{x}}\cdot$$

If the point y travels over the transversal (3.1), the point x will describe the locus

$$T^2x^2 + (\sigma_3 - \sigma_1T^2)x - \sigma_3^2\bar{x}^2 + (\sigma_2\sigma_3 - \sigma_3T^2)\bar{x} + \sigma_2T^2 - \sigma_1\sigma_3 = 0. \tag{4.1}$$

This is an equilateral hyperbola, also on the orthocenter H of $A_1A_2A_3$, with its center at

$$C = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right). \tag{4.2}$$

The equation of the tangent line at H to the hyperbola is

$$(\sigma_1T^2 + \sigma_3)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_2^2. \tag{4.3}$$

Now (4.2) is the point X_N, and (4.3) is the line of images of X_C of (3.2). Hence we can state the theorem *if any transversal be drawn through the circumcenter of $A_1A_2A_3$, cutting the sides A_2A_3, A_3A_1, A_1A_2 in the points C_1, C_2, and C_3 respectively, one point of intersection of the three circles on A_iC_i as diameters is the center of that equilateral hyperbola which is the isogonal conjugate of the given transversal, while the second point of intersection is the point on the circumcircle of $A_1A_2A_3$ whose line of images is the tangent at the orthocenter to the same hyperbola.*

The diametrically opposite point of H on the hyperbola (4.1), which lies also on the circumcircle of $A_1A_2A_3$, is

$$x = -\frac{\sigma_3}{T^2}\cdot \tag{4.4}$$

The line of images of (4.4) is

$$x + T^2\bar{x} = \sigma_1 + \frac{T^2\sigma_2}{\sigma_3},$$

which is perpendicular to (3.1). Hence *the fourth intersection of the equilateral hyperbola* (4.1) *with the circumcircle of $A_1A_2A_3$ is that point whose line of images is perpendicular to the given transversal.* We state without proof two other properties of the point (4.4). If we extend to the circumcircle, lines from the vertices of $A_1A_2A_3$ perpendicular to the transversal OT, and from these points draw perpendiculars to the opposite sides, the three perpendiculars meet at (4.4). The perpendiculars, from the two points of intersection of the transversal OT with the circumcircle of $A_1A_2A_3$, on their respective lines of images, intersect on the circumcircle at the point diametrically opposite to (4.4).

5. The Droz-Farny theorem. Any two perpendicular lines drawn through the orthocenter H of the triangle $A_1A_2A_3$ cut on the sides of the triangle three segments whose midpoints D_i lie on a line Δ. The perpendiculars through A_i to the lines HD_i intersect at a point P on the circumcircle; also PH is perpendicular to Δ.

We saw earlier that the lines of images of any two diametrically opposite points T and T' are perpendicular to each other at H. Using these as the transversals in the above theorem the coördinate of the point D_1 is found to be

$$d_1 = \frac{\sigma_3 + \sigma_1 T^2}{T^2 - t_1^2}\cdot \tag{5.1}$$

The three points D_i lie on the line

$$T^2(\sigma_2 + T^2)x + \sigma_3(\sigma_3 + \sigma_1 T^2)\bar{x} = (\sigma_2 + T^2)(\sigma_3 + \sigma_1 T^2). \tag{5.2}$$

The coördinate of P is $-\sigma_3/T^2$, while the equation of PH is

$$T^2(\sigma_2 + T^2)x - \sigma_3(\sigma_3 + \sigma_1 T^2)\bar{x} = \sigma_1 T^4 - \sigma_2\sigma_3. \tag{5.3}$$

Hence, *the point P of the Droz-Farny theorem is the fourth intersection of the circumcircle with that equilateral hyperbola which is the isogonal conjugate of the diameter TT'. Also, the line PH is the chord of contact of the three circles on A_iC_i as diameters where C_i are the intersections of the sides of $A_1A_2A_3$ with the transversal TOT'.*

The point P is the second intersection of the line X_CX_N with the circumcircle. Its line of images is perpendicular to TOT'. Further the lines through the points V_i of section 3, perpendicular to the sides of the triangle $A_1A_2A_3$, will meet on the circumcircle at a point P', diametrically opposite to P. Also, the parallels through H to the lines A_1P', A_2P', A_3P' meet the sides A_2A_3, A_3A_1, A_1A_2 in the points D_i and hence lie on the line Δ.

6. The point of Miquel. From any four non-specialized lines we can form four triangles, whose circumcircles all meet at a point μ called the point of Miquel of the complete quadrilateral. If we consider the quadrilateral formed by the sides of the triangle $A_1A_2A_3$ and the transversal (3.1), its point of Miquel is

$$\mu_3 = \frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1 T^2}\cdot \tag{6.1}$$

Similarly the coördinate of the point of Miquel for the quadrilateral formed by the sides of the triangle $A_1A_2A_3$ and the line (1.2) is

$$\mu_1 = \frac{\sigma_3 + \sigma_1 T^2}{\sigma_2 + T^2}\cdot \tag{6.2}$$

The points μ_1 and μ_3 are the points X_C of (3.2) and α of (3.4). Hence we can state the following theorems: *The point of Miquel for the quadrilateral formed by the sides A_1A_2, A_2A_3, A_3A_1 and any transversal OT through the circumcenter of*

$A_1A_2A_3$ is the image in OT of the point of intersection on the circumcircle of the circles on A_iC_i as diameters, where the C_i are the points of intersection of the transversal with the sides of $A_1A_2A_3$. The point of Miquel for the quadrilateral formed by the sides A_1A_2, A_2A_3, A_3A_1 and the line of images of a point T on the circumcircle of $A_1A_2A_3$ is likewise the point of intersection on the circumcircle of the circles on A_iC_i as diameters, where the C_i are the points of intersection with the sides of $A_1A_2A_3$ by the transversal joining T to the circumcenter of $A_1A_2A_3$. These two points of Miquel are the images of each other in the transversal OT.

We note from (6.2) that the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1 and the line of images of the point diametrically opposite to T on the circumcircle of $A_1A_2A_3$ will likewise be (6.2). Hence given any point μ on the circumcircle of $A_1A_2A_3$ to find the line which with A_1A_2, A_2A_3, A_3A_1 will have μ as its point of Miquel we proceed as follows: Join μ to the orthocenter H and find the point α whose line of images is μH. The perpendicular bisector of $\alpha\mu$ will cut the circumcircle in two points. The line of images of either point is the required line.

The line of images for the point of Miquel μ_1 is

$$(\sigma_3 + \sigma_1 T^2)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_2^2. \tag{6.3}$$

Let us find the points T for which the line (6.3) is parallel to OT. Equating the clinants of the lines we are led to $T^4 = \sigma_2\sigma_3/\sigma_1$. Let us find the points T for which the point of Miquel μ_3 will fall at Φ. Setting

$$\frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1 T^2} = \frac{\sigma_2}{\sigma_1},$$

we find $T^4 = \sigma_2\sigma_3/\sigma_1$. Since the line of images of μ_3 passes through μ_1, then μ_1 must be the intersection of the Euler line with the circumcircle and the points T are the intersections with the circumcircle of the perpendicular bisector of $\mu_1\mu_3$. Either point is such that the point of Miquel, for the quadrilateral A_1A_2, A_2A_3, A_3A_1 and the image line of T, has its image line parallel to OT, while the point of Miquel for the quadrilateral A_1A_2, A_2A_3, A_3A_1, and OT is Φ.

Let us find the points T for which (6.3) is the join of the orthocenter with the diametrically opposite point of T. This latter line has the equation

$$(\sigma_3 + \sigma_2 T)x - \sigma_3 T(\sigma_1 + T)\bar{x} = \sigma_1\sigma_3 - \sigma_2 T^2. \tag{6.4}$$

For (6.3) and (6.4) to be identical we are led to

$$\sigma_1 T^4 + (\sigma_1^2 - \sigma_2)T^3 + (\sigma_1\sigma_3 - \sigma_2^2)T - \sigma_2\sigma_3 = 0.$$

whose four roots are $T = \sigma_2/\sigma_1$, $-t_1$, $-t_2$, $-t_3$. Hence *the four points, Φ and the diametrically opposite points of A_1, A_2, and A_3, are the only points whose lines of images form quadrilaterals with A_1A_2, A_2A_3, A_3A_1 such that the image line of their point of Miquel is the join of the orthocenter with their diametrically opposite point.*

7. The orthopole. If we drop perpendiculars from the vertices of a triangle $A_1A_2A_3$ upon any line, the perpendiculars from their feet are concurrent at a

point called the orthopole of the line as to $A_1A_2A_3$ [7]. Now the orthopole of the transversal OT as to $A_1A_2A_3$ is

$$x = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right) = X_N, \tag{7.1}$$

while the orthopole of (1.2), the image line of T, as to $A_1A_2A_3$ is the point

$$\beta = \sigma_1 - \frac{1}{2}\left(T + \frac{\sigma_2}{T}\right). \tag{7.2}$$

The locus of β as the point T traverses the circumcircle of $A_1A_2A_3$ is an ellipse with center at H, the orthocenter of $A_1A_2A_3$. It intersects the nine-point circle at the feet of the altitudes of $A_1A_2A_3$ and at the center of Jerebek's hyperbola. The equations of its axes are

$$\sqrt{\frac{\sigma_1\sigma_3}{\sigma_2}}\, x \pm \sigma_3\bar{x} = \sigma_1\sqrt{\frac{\sigma_1\sigma_3}{\sigma_2}} \pm \sigma_2.$$

These are the lines of images of the two intersections of the Euler line with the circumcircle, and hence can easily be constructed.

The image of the point β in the line (1.2) is

$$x = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right) = X_N.$$

We have proved the two theorems: *The orthopole of the line joining any point T on the circumcircle of $A_1A_2A_3$ to the circumcenter O is that point of intersection on the nine-point circle of the three circles on A_iC_i as diameters where C_i are the points of intersection on the sides of $A_1A_2A_3$ with the line joining T to the circumcenter of $A_1A_2A_3$. The orthopoles as to $A_1A_2A_3$ of the lines* (1.2) *and* (3.1) *are images of each other in the line* (1.2).

8. Some special cases. To illustrate the relationships shown in this paper, we state some particular cases, recalling that the isogonal conjugate of the Euler line is the hyperbola of Jerebek, while that of the Brocard line is Kiepert's hyperbola.

If the Euler line of the triangle $A_1A_2A_3$ cuts the sides A_2A_3, A_3A_1, and A_1A_2 in the points C_1, C_2, and C_3 respectively, the three circles on A_iC_i as diameters meet in two points—one the center of Jerebek's hyperbola, the other the image in the Euler line of the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1, and the Euler line. If we reflect any point on the common chord of the three circles in the sides of the triangle $A_1A_2A_3$, obtaining the points B_1, B_2, and B_3 the three circles $B_1B_2A_3$, $B_2B_3A_1$, and $B_3B_1A_2$ meet at the point of Miquel for the quadrilateral formed by the lines A_1A_2, A_2A_3, A_3A_1, and the Euler line. The center of Jerebek's hyperbola is also the orthopole of the Euler line as to the triangle $A_1A_2A_3$.

If the Brocard line of the triangle $A_1A_2A_3$ cuts the sides of the triangle in

points C_1, C_2, and C_3, the three circles on A_iC_i as diameters meet at the center of the Kiepert hyperbola and at the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1 and the line joining the orthocenter and the symmedian point of $A_1A_2A_3$. The orthopole of the Brocard line as to the triangle $A_1A_2A_3$ is the center of the Kiepert hyperbola.

If we use as the transversal the line joining the circumcenter to the point Φ, the three circles meet at the image in the Euler line of the center of Jerebek's hyperbola and at the point of Miquel for the quadrilateral formed by the lines A_1A_2, A_2A_3, A_3A_1, and the Euler line.

The fourth intersection of the hyperbola of Jerebek with the circumcircle is the diametrically opposite point on the circumcircle of Φ, the point of Feuerbach of the tangential triangle of $A_1A_2A_3$.

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THE NINE CIRCLE THEOREM AND THE ENLARGED GEOMETRY

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Here we state the Nine Circle Theorem in the following two forms:

(a). The *elementary* form: Let A_i $(i=1, 2, 3)$ be any three circles which have six external common tangents. If we construct the circles D_i $(i=1, 2, 3)$ such that D_i is tangent to the external common tangents of A_j and A_k $(i, j, k=1, 2, 3; j\neq k, k\neq i, i\neq j)$ at the points, through which the line is the polar of some point on the axis of the external centers of similitude of the given circles with respect to D_i, it may be proved that the six external common tangents of A_i and D_i $(i=1, 2, 3)$ will be tangent to a fixed circle. Further, let H be this circle, and construct the six circles E_i and F_i $(i=1, 2, 3)$ such that E_i is tangent to the external common tangents of A_j and A_k at their middle points and F_i to those of A_i and H at their middle points. Then each of the nine circles D_i, E_i and F_i $(i=1, 2, 3)$ will be tangent to two fixed circles.

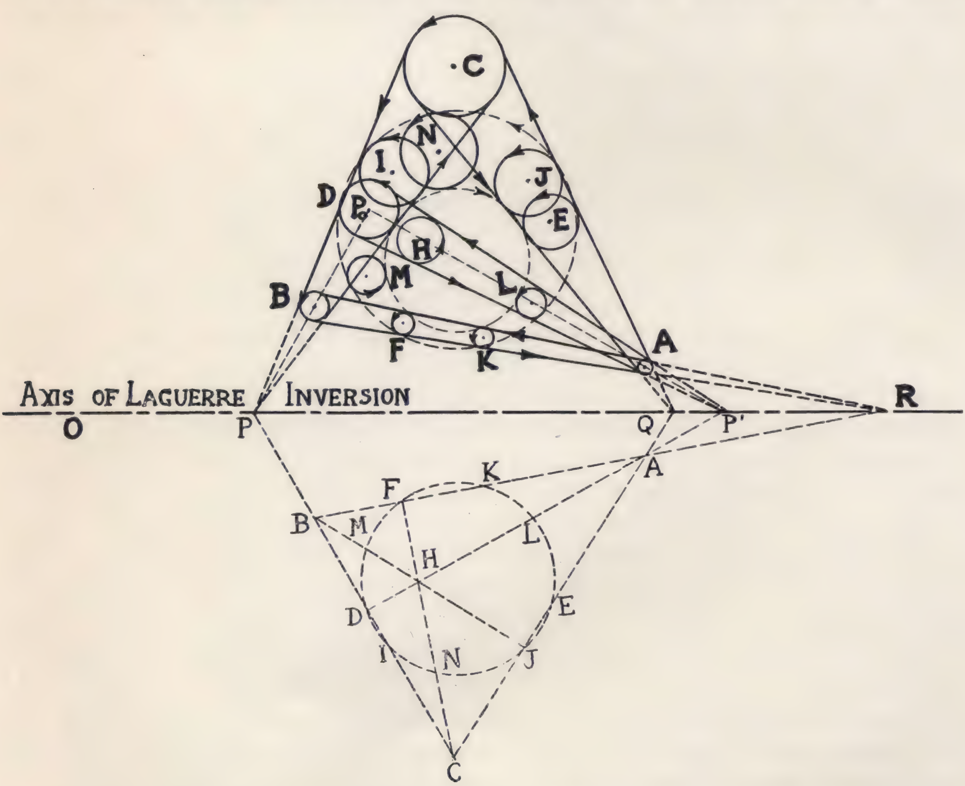
(b). The *oriented* form: Let **A**, **B**, and **C** be any three oriented circles, which have six common proper tangents. Construct the oriented circle **D**, such that **D** is properly tangent to the common proper tangents of **B** and **C**, and the centers of similitude of **B** and **C**, **A** and **D**, form a pair of conjugate points with respect to **D**. Similarly, we may construct the oriented circles **E** and **F** by replacing **ABCD** by **BCAE** and **CABF** respectively in the above construction. At this stage it may be proved that the six common proper tangents of **A** and **D**, **B** and

E, **C** and **F**, will be properly tangent to a fixed oriented circle. Call it **H**, and construct the six oriented circles **I**—**N** properly tangent to, and passing through the middle points of, the common proper tangents of the six pairs **B** and **C**, **C** and **A**, **A** and **B**, **A** and **H**, **B** and **H**, **C** and **H** respectively. Then each of the nine oriented circles **D**—**F** and **I**—**N** will be properly tangent to two fixed oriented circles.

It is evident that we have only to prove our theorem in the latter form. Accordingly we give the following:

Proof. For the sake of simplicity we prove our theorem by Laguerre inversion.*

Let O be the axis of similitude of the three given oriented circles, and take O as the axis of a Laguerre inversion. Let A be one of the two point circles, which



together with **A** have the same radical axis O. Take the constant of the Laguerre inversion such that **A** is inverted into A. Let P, Q, R be the centers of similitude of the pairs **B** and **C**, **C** and **A**, **A** and **B** respectively. Each of them is then inverted into itself. Since **B** is an oriented circle, which is properly tangent to the common proper tangents of **A** and R, **B** will be inverted into a unique oriented

* Laguerre, Oeuvres, Paris, 1905, vol. 2, pp. 592–619.

circle B, which is properly tangent to the common proper tangents of the point circles A and R, and is therefore a point circle. Similarly the oriented circles $\mathbf{C}-\mathbf{F}$ are inverted into unique point circles, which are denoted by $C-F$ respectively.

The center of similitude P' of the oriented circles $\mathbf{A}$ and $\mathbf{D}$ is then a point of O, for A and D, the Laguerre inverses of $\mathbf{A}$ and $\mathbf{D}$, are point circles, so that P' will be inverted into itself and lie on O. By hypothesis the pair of points P and P' are conjugate with respect to $\mathbf{D}$. Therefore P' is the point of intersection of O and the polar of P with respect to $\mathbf{D}$. Now let P_0 be the center of $\mathbf{D}$, d the radius of $\mathbf{D}$, and t and t' the lengths of the tangents to $\mathbf{D}$ from P and P' respectively, if neither P nor P' is at infinity. Then since PP_0 is perpendicular to the polar of P with respect to $\mathbf{D}$, we have

$$PP'^2 - P_0P'^2 = t^2 - d^2.$$

But $P_0P'^2 = t'^2 + d^2$, and therefore $t^2 + t'^2 = PP'^2$, which shows that in the inverted figure we must have

$$PD^2 + P'D^2 = PP'^2.$$

Hence D is the foot of the perpendicular from A to BC, which proposition may easily be proved also to be true when P or P' is at infinity. Similarly E and F are the feet of perpendiculars from B and C to CA and AB respectively.

Now by an elementary property of the triangle ABC, AD, BE, and CF are concurrent; *i.e.*, the six oriented lines AD, DA, BE, EB, CF, and FC are properly tangent to the same point circle H. Hence the six proper tangents of $\mathbf{A}$ and $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{E}$, $\mathbf{C}$ and $\mathbf{F}$ will be properly tangent to the same oriented circle $\mathbf{H}$, the Laguerre inverse of H. This completes the proof of the first part of our theorem.

Further, by hypothesis, it is evident that the oriented circles $\mathbf{I}-\mathbf{N}$ are inverted into the middle points $I-N$ of BC, CA, AB, AH, BH, and CH respectively. But from the ordinary property of the nine point circle of the triangle ABC we see that the nine points $D-F$ and $I-N$ are concyclic; *i.e.*, each of the nine point circles $D-F$ and $I-N$ is properly tangent to some two fixed oriented circles. Hence each of their Laguerre inverses, the nine oriented circles $\mathbf{D}-\mathbf{F}$ and $\mathbf{I}-\mathbf{N}$, will also be properly tangent to some other two fixed oriented circles. Q.E.D.

Note 1. If we assume that the triangle ABC is real, we have only to add the necessary and sufficient condition that none of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ contains a real point on their axis of similitude.

Note 2. The Nine Circle Theorem here and the Nine Point Circle Theorem in text books are so related that each of their figures is the Laguerre inverse of the other. Hence the truth of one theorem insures the truth of the other.

Note 3. In general each figure in a euclidean plane has a Laguerre inverse, so that a large number of new theorems concerning oriented circles can be deduced from euclidean theorems by Laguerre inversion.

Note 4. We may think of Laguerre inversion as an enlarger and define any

oriented circle, which is obtained from an ordinary point by a Laguerre inversion, as an enlarged point, which, not being ordinary, may be said to have not only position but also form and size. Then the length of any one of the common proper tangents of two oriented circles, the Laguerre inverse of the distance of two ordinary points, may be defined as the distance between the two corresponding enlarged points. A pair of oriented lines, being the Laguerre inverse of a line, is defined as an enlarged line, and any oriented circle which is properly tangent to them as an enlarged point of the enlarged line. A pair of oriented circles, being the Laguerre inverse of a circle, is defined as an enlarged circle, and any oriented circle which is properly tangent to them as an enlarged point of the enlarged circle. A pair of two enlarged lines is defined as parallel if the oriented lines of the first enlarged line are respectively parallel to those of the second. On the other hand a pair of two enlarged lines is defined as intersecting, if their oriented lines are all properly tangent to the same oriented circle. Lastly, a pair of two intersecting enlarged lines is defined as perpendicular, if the point of intersection of the oriented lines of the first enlarged line is conjugate to that of the second with respect to the enlarged point of intersection. The Nine Circle Theorem may then be stated in the following form (the enlarged form):

The three enlarged perpendiculars from the enlarged vertices of an enlarged triangle **ABC** to the opposite enlarged sides have an enlarged point in common. Call it **H**. Then the enlarged feet **D**, **E**, **F** of the enlarged perpendiculars and the enlarged middle points of **BC**, **CA**, **AB**, **AH**, **BH**, **CH** lie on the same enlarged circle.

Note 5. From the Notes 3 and 4 we obtain at once the following theorem, which may be called:

The Principle of Enlargement. From each euclidean theorem we can always deduce a new theorem by changing the ordinary conceptions, point, line, circle into the enlarged ones, enlarged point, enlarged line, enlarged circle respectively.

By means of this principle we obtain a new geometry, called the Enlarged Geometry, which contains the theorems of enlarged elements. The Nine Circle Theorem is one of such theorems if it is written in the enlarged form in Note 4. It becomes an application of this geometry if it is changed into the forms (a) and (b) by replacing the enlarged elements by their ordinary equivalents.

Note by the Editor. Clearly any transformation applied to a configuration in one theorem leads to a (new) theorem regarding a (new) configuration. The Laguerre inversion is a transformation known for over fifty years which leads to interesting, though somewhat complicated, results, as shown in this paper. This transformation seems to have received little attention, Laguerre receiving no mention, for example, in Morley's *Inversive geometry*.

THE SIMSON QUARTIC OF A TRIANGLE

C. E. VAN HORN, Fisk University

1. Introduction. The Simson line of any point on the circumcircle of a triangle is the line containing the feet of the perpendiculars from the point to the sides of the triangle. The existence of a line thus related to the circumcircle was discovered* by Robert Simson (1687–1768). It is also called the *pedal line* of the point with reference to the given triangle.

By the Simson quartic of a triangle ABC is meant the envelope of its Simson lines. The existence of this envelope as a curve of order four and class three was discovered by Steiner† in 1857, and its character as a three-cusped hypocycloid circumscribing the triangle demonstrated by Cremona‡ in 1865.

The properties that form the subject matter of this paper are the following: First it will be shown that the Simson quartic of a triangle is a three-cusped hypocycloid with its center at the nine-point center of the triangle,§ the hypocycloid being generated by a circle with a radius equal to that of the nine-point circle rolling on the inside of a circle with a radius which is three times that of the nine-point circle. This is Cremona's theorem, already referred to. The second property gives rise to a simple construction for determining the position of the cuspidal tangents of the Simson quartic of a triangle. The third property is the theorem that the points where the Simson quartic of a triangle touches its sides are the symmetrics of the feet of the altitudes relative to the midpoints of the respective sides. Finally the theorem is proved that the altitudes of a triangle touch its Simson quartic, each altitude touching the quartic at a point whose distance from the foot of the altitude is twice the projection on it of the circumradius to the vertex through which the altitude is drawn. The specifically detailed relationships of the Simson quartic of a triangle to that triangle given in the last three properties do not appear to have been explicitly stated before.

A simple construction of the cuspidal tangents of the Simson quartic of a

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* *Note by Editor*. This is debatable. See R. A. Johnson, Modern Geometry, p. 137. J. R. M.

† J. Steiner, Über eine besondere Curve dritter Klasse (und vierten Grades), Crelle, vol. 53, 1857, p. 231.

‡ Cremona, Sur l'hypocycloide à trois rebroussements, Crelle, vol. 64, 1865, p. 101.

§ The nine-point circle of a triangle is so named because it passes through the three midpoints of the sides of the triangle, the three feet of its altitudes, and the three midpoints of the segments of the altitudes joining the orthocenter of the triangle to its vertices. A theorem due to Euler states that the circumcenter, the centroid, the nine-point center, and the orthocenter of a triangle lie on one line, called *the Euler line* of the triangle in question. The radius of the nine-point circle is one half that of the circumcircle, and its center is the midpoint of the segment of the Euler line between the circumcenter and orthocenter.

If a circle of radius b rolls without slipping on the inside of a circle of radius a, $(a>b)$, a point of the first traces out a curve called a hypocycloid in the common plane of the two circles. If a equals $3b$, the resulting hypocycloid is the familiar three-cusped hypocycloid of elementary texts on the calculus, whose parametric equations are:

$$(1)\qquad \begin{aligned} x &= 2b\cos\theta + b\cos(2\theta - 3\alpha),\\ y &= 2b\sin\theta - b\sin(2\theta - 3\alpha). \end{aligned}$$

triangle exists, based on a theorem in elementary geometry that seems to have escaped explicit statement.* It is stated most readily by reference to the accompanying figure; the proof, a simple exercise in elementary geometry, is omitted.

Let ABC be any triangle and let O be its circumcenter. Let D, E, F be the midpoints of the sides BC, AC, and AB respectively. Draw the side bisector DH of the side BC to cut the arc BAC of the circumcircle containing the opposite vertex A at the point H. The side bisectors EJ and FK are similarly drawn to cut the arcs ABC and BCA at the points J and K respectively. Choose the point L on the arc HA so that HL is one third of the arc HA. Select the points M and N in a similar manner on the arcs JB and KC respectively. The theorem in question is that whatever triangle ABC may be, the triangle LMN is equilateral. It will be referred to as *the equilateral derivative* of the triangle ABC.

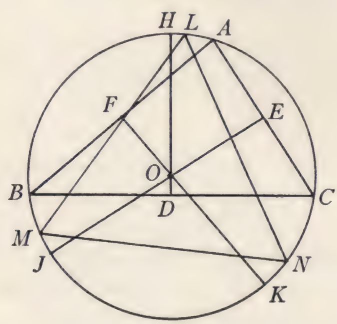

FIG. 1

2. The four theorems. Let ABC be any triangle (Figure 2), and let it be referred to its circumcenter O as origin, the side bisector of AC as axis of x and the line through O parallel to AC as axis of y. The radius of the circumcircle of ABC shall be the unit of length.

Let the vertices of the triangle ABC be the points: $A \equiv (\cos 2\alpha, \sin 2\alpha)$, $B \equiv (\cos 6\beta, \sin 6\beta)$, $C \equiv (\cos 2\alpha, -\sin 2\alpha)$. There is no loss of generality in this asymmetrical orientation of the triangle. The orthocenter R is the point $(2 \cos 2\alpha + \cos 6\beta, \sin 6\beta)$. The nine-point center Q has the coördinates $(\cos 2\alpha + \frac{1}{2} \cos 6\beta, \frac{1}{2} \sin 6\beta)$. The radius of the nine-point circle of ABC is $1/2$. By assumption OEJ is the side bisector of AC and the axis of x. If the arc JM is $1/3$ of the arc JB, then M is a vertex of the equilateral derivative of ABC and has the coördinates$(\cos 2\beta, \sin 2\beta)$. The line OM with the equation $y = x \tan 2\beta$ is one of the altitudes of the equilateral derivative of ABC, since obviously the circumcenter of ABC is also the circumcenter of its equilateral derivative.

Let $P \equiv (\cos 2\phi, \sin 2\phi)$ be any point on the circumcircle of the triangle ABC. The equation of its Simson line STU relative to ABC is

* *Note by Editor.* The theorem was proved but not explicitly stated by Goormagtigh, Mathesis, vol. 31, 1911, p. 163. J.R.M.

$$y - \sin 2\phi = \tan (3\beta - \phi)(x - \cos 2\alpha). \tag{2}$$

If we differentiate (2) with respect to ϕ, and solve the resulting equation and (2) for x and y in terms of ϕ we obtain the parametric equations of the envelope of the Simson lines of the triangle ABC, namely,

$$\begin{cases} x = \cos 2\alpha + \frac{1}{2} \cos 6\beta + \cos 2\phi + \frac{1}{2} \cos (4\phi - 6\beta), \\ y = \qquad\qquad \frac{1}{2} \sin 6\beta + \sin 2\phi - \frac{1}{2} \sin (4\phi - 6\beta). \end{cases} \tag{3}$$

When referred to the nine-point center Q of the triangle and to axes parallel to the original axes the parametric equations of the envelope become

$$\begin{cases} x = \cos 2\phi + \frac{1}{2} \cos (4\phi - 6\beta), \\ y = \sin 2\phi - \frac{1}{2} \sin (4\phi - 6\beta). \end{cases} \tag{4}$$

A comparison of equations (4) with equations (1) yields the following result, remembering that the radius of the nine-point circle of the triangle ABC is 1/2.

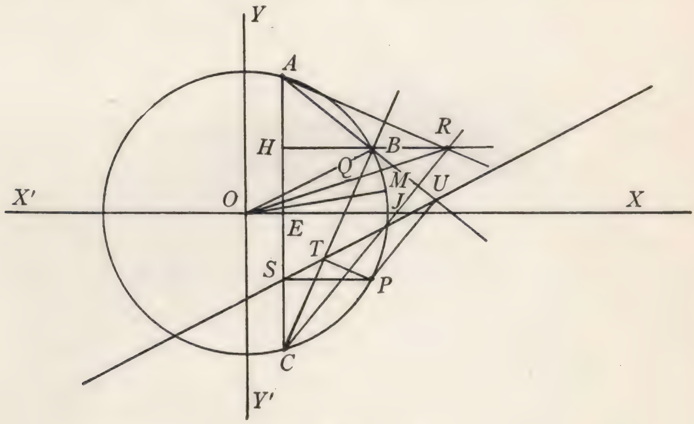

FIG. 2

THEOREM I. *The envelope of the Simson lines of the triangle ABC is a three-cusped hypocycloid generated by a circle equal in size to the nine-point circle of ABC rolling on the inside of a circle of radius three times the radius of, and with center at the center of, the nine-point circle of ABC.*

The line joining a cusp of a three-cusped hypocycloid to the center of its fixed circle is one of the cuspidal tangents possessed by the curve. Moreover since it makes the angle 2β with the x-axis it is parallel to OM of Figure 2. As OM is an altitude of the equilateral derivative of ABC, it follows that the cuspidal tangent in question is perpendicular to a side of the equilateral derivative of the triangle ABC. Since B may be any vertex and AC its opposite side, we have the result:

THEOREM II. *The cuspidal tangents of the Simson quartic of any triangle are perpendiculars through its nine-point center to the sides of its equilateral derivative.*

But this is not all. The slope of the cuspidal tangent in question is $\tan 2\beta$ and has the equation

$$y - \sin 2\beta = \tan 2\beta(x - \cos 2\alpha), \tag{5}$$

since it passes through the nine-point center. It is readily verified that it is the Simson line of the point $M \equiv (\cos 2\beta, \sin 2\beta)$, a vertex of the equilateral derivative of the triangle ABC. Hence it follows as a corollary to Theorem II that *the vertices of the equilateral derivative of any triangle have as Simson lines the cuspidal tangents of its Simson quartic.*

Obviously the location of the cuspidal tangents by means of the equilateral derivative of the triangle ABC locates also the points where its Simson quartic touches its nine-point circle.

The equation of the altitude through the vertex B to the side AC is $y = \sin 6\beta$. Its foot on AC is the point H with the coördinates $(\cos 2\alpha, \sin 6\beta)$. The side AC is the Simson line of the point $(-\cos 6\beta, -\sin 6\beta)$. The point where AC touches the Simson quartic of ABC is $(\cos 2\alpha, -\sin 6\beta)$, the symmetric of the foot of the altitude on AC relative to the midpoint of AC. As B may be any vertex and AC its opposite side, we have:

THEOREM III. *The Simson quartic of any triangle touches each of its sides at the symmetric of the foot of the altitude on that side relative to its midpoint.*

The Simson line of the vertex $B \equiv (\cos 6\beta, \sin 6\beta)$ is the altitude $y = \sin 6\beta$. Its point of contact with the Simson quartic of ABC is $(\cos 2\alpha + 2 \cos 6\beta, \sin 6\beta)$. The distance of this point of contact from the foot of the altitude is $2 \cos 6\beta$, twice the projection of the circumradius to B on the altitude from B to AC. Hence we have the result:

THEOREM IV. *The altitudes of any triangle touch its Simson quartic, each altitude touching the quartic at a point whose distance from its foot is twice the projection on it of the circumradius to the vertex through which the altitude is drawn.*

A NOTE ON FERMAT'S LAST THEOREM

DAVID SEGAL, Kosov-Huculski, Poland

1. A theorem of Peter Barlow* states that in case

$$x^n + y^n + z^n = 0 \tag{1}$$

has a solution (x, y, z) in coprime integers prime to the odd prime n, then integers r, s, t exist such that

$$x + y = r^n, \tag{2}$$

$$y + z = s^n, \tag{3}$$

$$z + x = t^n, \tag{4}$$

In order to obtain lower limits for x, y, and z in (1) G. James† has had to show that

$$r + s + t \neq 0. \tag{5}$$

His proof of this interesting fact is a lengthy geometrical argument. The purpose of this note is to show how a number-theoretic proof of (5) may be given in a few lines.

In the first place since x^n+y^n is algebraically divisible by $x+y$ when n is odd, it follows from (1) and (2) that r^n divides z^n so that

$$z = ru. \tag{6}$$

Similarly

$$x = sv \tag{7}$$

and

$$y = tw. \tag{8}$$

Now suppose, if possible, that (5) is false so that

$$r + s + t = 0; \tag{9}$$

then it is seen that precisely one of (r, s, t) is even. In fact all three could not be even for in this case, in view of (6), (7), and (8), the same would be true of the supposedly coprime (x, y, z). Hence we may suppose by symmetry that r is even and s and t are odd.

Using (2), (3), (4), and (6) we may write

$$s^n + t^n = x + y + 2z = r^n + 2ru. \tag{10}$$

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* See for instance Dickson's History of the Theory of Numbers, vol. 2, p. 733, where the simple argument needed to establish the result is given. In the present note the notation has been slightly altered to render the problem completely symmetric.

† This MONTHLY, vol. 41, 1934, pp. 419–424

Dividing both members by r we have in view of (9)

$$-(s^n + t^n)/(s + t) = r^{n-1} + 2u. \tag{11}$$

Now the right member of (11) is even since r is even. But the left member is odd since it can be written as a sum

$$-(s^{n-1} - s^{n-2}t + - \cdots + t^{n-1})$$

of n odd numbers. This contradiction proves (5).

2. The limits set by James for x, y, and z can be raised very easily. A short proof can be given of the fact that each of r, s, and t must be greater than or equal to $2n^2+1$. If we suppose, for definiteness, that $x>y>z$, it follows from (1) that

$$(2n^2 + 1)^n < r^n = x + y < z.$$

Since n is known to be greater than 14000, we see that the smallest possible value of x, y, or z exceeds

$$(392000000)^{14000} > 10^{10^5}.$$

A HIGHER UPPER LIMIT TO THE PARAMETERS IN FERMAT'S EQUATION*

GLENN JAMES, University of California at Los Angeles

1. Introduction. This paper enlarges the ranges of x, y, and z on which the Fermat equation with x, y, and z not divisible by n, the so-called "first case," has no solution. We consider the Fermat equation

$$x^n + y^n = z^n \tag{1}$$

where x, y, and z are positive integers, $n>2$, and $x, y, z\not\equiv 0 \pmod{n}$. As is well known it suffices to treat only those cases in which n is an odd prime and x, y, and z are relatively prime.

In a previous paper† I proved that this equation has no solution if the least of x, y, z is less than $n(2cn+1)^n$. Equation (7) of that paper states that if equation (1) has a solution, (x, y, z), then

$$(x/n) > z - y \tag{2}$$

where

$$x < y < z,$$

and Theorem II of that paper states that

* This is a combination of two papers, which were presented to the American Mathematical Society, November 28, 1936 and November 29, 1937.

† This MONTHLY, vol. 41, 1934, p. 421.

$$z - y \geqq (2cn + 1)^n \tag{3}$$

where

$$2cn = (z - y)^{1/n} + (z - x)^{1/n} - (x + y)^{1/n}, \tag{4}$$

and c is a positive integer.

The early part of the present paper obtains the identity, (4), by a method which shows that c contains n. Thereby the limit for $z-y$ given in inequality (3) is stepped up to $(2cn^2+1)^n$. And consequently the limit for x is, by relation (2), increased to $n\ (2cn^2+1)^n$.

The latter part of this paper raises this new limit to $(111/77)n(2cn^2+1)^n$, by means of a certain interpolation process.

2. Some basic theorems. The work that follows is based on a theorem by Peter Barlow* and a related theorem credited to Sophie Germain by Legendre.† We give brief proofs of both these important theorems.

The Barlow Theorem states that *if x, y, z satisfy the Fermat equation, are relatively prime, and x, y, $z \not\equiv 0$ (mod n) then $z-x$, $z-y$, and $x+y$ are nth powers of integers.*

To prove this we start with the identity

$$z^n = (y + \overline{z - y})^n,$$

expand the right member, transpose the term, y^n, replace $z^n - y^n$ by x^n and factor $z-y$ out of the terms on the right. This gives

$$\begin{aligned} x^n = (z - y)\big[ny^{n-1} + \tfrac{1}{2}n(n-1)y^{n-2}(z-y)^1 + \cdots \\ + ny(z-y)^{n-2} + (z-y)^{n-1}\big]. \end{aligned} \tag{5}$$

Now $z-y$ and the bracket are relatively prime, for $z-y$ and y are relatively prime since z and y are, by hypothesis, and also $z-y$ cannot contain n else x would. It follows that every prime factor of $z-y$ appears to the (kn)th power in $z-y$ since it appears to this power in x^n. Thus $z-y$ is the nth power of some integer. Similar arguments establish the same property for $z-x$ and $x+y$.

The Sophie Germain Theorem may be stated as follows.

If x, y, z satisfy the Fermat equation, are relatively prime, and x, y, $z \not\equiv 0$ (mod n), then

$$\begin{aligned} &\text{(a)} \quad x = (z - y)^{1/n}(2c_1n^2 + 1), \\ &\text{(b)} \quad y = (z - x)^{1/n}(2c_2n^2 + 1), \\ &\text{(c)} \quad z = (x + y)^{1/n}(2c_3n^2 + 1). \end{aligned} \tag{6}$$

* The Journal of Natural Philosophy, Chemistry and Arts, vol. 27, 1810, p. 193.

† Legendre, Théorie des Nombres, ed. 2, 1808, second supplement, September, 1825, pp. 1–40.

Proof: Equation (5) can be written

$$x^n = (z - y)(2d_1n + 1); \tag{7}$$

for by Fermat's Simple Theorem,

$$(z - y)^{n-1} \equiv 1 \pmod{n},$$

and the bracket in equation (5) is odd since two and two only of x, y, and z must be odd and the bracket and $z-y$ are relatively prime as we have just proven.

Again applying the Fermat Simple Theorem we have

$$x^n \equiv x \pmod{n},$$

and

$$[(z - y)^{1/n}]^n \equiv (z - y)^{1/n} \pmod{n}.$$

By means of these reductions, equation (7) can be written

$$\text{(a}'\text{)} \quad x = (z - y)^{1/n}(2e_1n + 1). \tag{8}$$

And we obtain similarly the relations,

$$\text{(b}'\text{)} \quad y = (z - x)^{1/n}(2e_2n + 1),$$

$$\text{(c}'\text{)} \quad z = (x + y)^{1/n}(2e_3n + 1).$$

Now let α be any prime factor of $(2e_1n+1)$, that is, of $x/(z-y)^{1/n}$. Obviously α is odd and prime to n, $z-x$, $z-y$, and $x+y$.

Now we have

$$(x + y)^n - (z - x)^n \equiv 0 \pmod{\alpha} \tag{9}$$

since

$$(x + y)^n - (z - x)^n = x^n + y^n - z^n + \text{terms in } x. \tag{10}$$

In order to simplify congruence (9) we note that

$$[(z - x)^{1/n}]^{\alpha-1} \equiv 1 \pmod{\alpha}. \tag{11}$$

Raise this congruence to the power n^2 and write it in the form

$$(z - x)^n(z - x)^{n(\alpha-2)} \equiv 1 \pmod{\alpha}. \tag{12}$$

Now multiply congruence (9) by $(z-x)^{n(\alpha-2)}$ and apply congruence (12) to the second term. This leaves

$$(z - x)^{n(\alpha-2)}(x + y)^n \equiv 1 \pmod{\alpha},$$

which can be written

$$[(z - x)^{(\alpha-2)/n}(x + y)^{1/n}]^{n^2} \equiv 1 \pmod{\alpha}. \tag{13}$$

The number, n^2, is the least exponent which applied to this bracket gives a result congruent to unity, modulus α. For if there were a lesser one it must, by

elementary number theory, be a factor of all higher powers of the bracket for which the congruence holds, that is, it must be either unity or n. But if it were unity then

$$(z - x)^{(\alpha-2)/n}(x + y)^{1/n} \equiv 1 \pmod{\alpha},$$

which taken with congruence (11) would give

$$(x + y)^{1/n} \equiv (z - x)^{1/n} \pmod{\alpha}.$$

This congruence raised to the nth power gives

$$x + y \equiv z - x \pmod{\alpha}, \tag{14}$$

or

$$2x \equiv z - y \pmod{\alpha},$$

which is impossible since α is a factor of x but prime to $z-y$. Similarly if the least exponent possible in congruence (13) were n, we would have

$$(z - x)^{\alpha-2}(x + y) \equiv 1 \pmod{\alpha},$$

which taken with the nth power of congruence (11) would give us again the impossible relation (14).

Now, since α is prime to both $z-x$ and $x+y$, we have

$$[(z - x)^{(\alpha-2)/n}(x - y)^{1/n}]^{\alpha-1} \equiv 1 \pmod{\alpha}.$$

Hence $(\alpha-1)$ is a multiple of n^2, that is

$$\alpha - 1 = b_1 n^2,$$

or since α is odd

$$\alpha = 1 + 2c_1 n^2.$$

From this result, identity (a) of (6) follows. Relations (6), (b) and (c) are proved similarly.

3. Derivation of the limit, $n(2cn^2+1)^n$. We write, from equations (6), the congruences,

$$\begin{aligned} &(\text{a}'') && (z - y)^{1/n} \equiv x && \pmod{2n^2},\\ &(\text{b}'') && (z - x)^{1/n} \equiv y && \pmod{2n^2},\\ &(\text{c}'') && -(x + y)^{1/n} \equiv -z && \pmod{2n^2}.\end{aligned}$$

It follows that

$$(z - y)^{1/n} + (z - x)^{1/n} - (x + y)^{1/n} \equiv x + y - z \pmod{2n^2}. \tag{15}$$

Also, raising each member of the congruences (a''), (b''), and (c'') to the nth power and adding, we obtain

$$x + y - z \equiv 0 \pmod{2n^3}, \tag{16}$$

since exactly two of x, y, z are odd.

Combining congruences (15) and (16) gives

$$(z - x)^{1/n} + (z - y)^{1/n} - (x + y)^{1/n} \equiv 0 \pmod{2n^2}. \tag{17}$$

By relation, (4), the left member of this congruence is positive. Hence we may write

$$(z - y)^{1/n} = 2cn^2 + (x + y)^{1/n} - (z - x)^{1/n},$$

where c is a positive integer. And since $(z-x)<(x+y)$ we have

$$(z - y) \geqq (2cn^2 + 1)^n. \tag{18}$$

Finally relation (18) taken with inequality (2) gives the important conclusion

$$x \geqq n(2cn^2 + 1)^n. \tag{19}$$

Since this, the so-called "first case" of the Fermat equation, is known to have no solution for $n \leqq 14000$, it follows, from the limit given by inequality (19), that it has no solution for any n if x, y, or z is less than 10^{120310}. The corresponding former limit was 10^{62200}. (Incidentally we are considering here only the highest degree terms in the expansions of the binomials $(2n^2+1)^n$ and $(2n+1)^n$.

4. A further increase in the lower limit of possible solutions of the Fermat equation.

The procedure here consists of setting up two inequalities between x, y, and $z-y$, then iterating between them to secure a larger, lower limit for x in terms of the known, lower limit of $z-y$.

The first equation is derived from the identity

$$y - x = (z - x) - (z - y). \tag{20}$$

Since by hypothesis

$$x < y < z,$$

we have

$$(z - y)^{1/n} + 1 \leqq (z - x)^{1/n}.$$

Applying this to (20) we may write

$$y - x \geqq [(z - y)^{1/n} + 1]^n - (z - y) \geqq n(z - y)^{(n-1)/n}, \tag{21}$$

whence

$$y \geqq x + n(z - y)^{(n-1)/n} \geqq x + na^{n-1}, \tag{22}$$

where a^n is a constant for a given n and is equal to or less than $(z-y)$ for all z and y which could possibly satisfy equation (1) for the given n.

The second relation needed is obtained from the identity

$$z^n = (y + \overline{z - y})^n.$$

Subtracting y^n from both sides and replacing $(z^n - y^n)$ by x^n enables us to write

$$x = y[(1 + (z - y)/y)^n - 1]^{1/n} \geqq y[(1 + a^n/y)^n - 1]^{1/n}. \tag{23}$$

We now return to relation (22), replace x by na^n, in view of equation (2), and substitute the resulting right member for y in equation (23). This gives

$$x \geqq n(a^n + a^{n-1})[(1 + a/[n(a + 1)])^n - 1]^{1/n}. \tag{24}$$

This step is justified by the fact that the right member of equation (23) is an increasing function of y.

From inequality (24) we can conclude that

$$x \geqq n(a^n + a^{n-1}), \tag{25}$$

provided the bracket in relation (24) is equal to or greater than unity.* For the present we shall assume this to be true and proceed with our iterating. Substituting the right member of inequality (25) for x in relation (22) we obtain

$$y \geqq n(a^n + 2a^{n-1}). \tag{26}$$

Substituting the right member of this inequality in place of y in equation (23) gives

$$x \geqq n(a^n + 2a^{n-1})[(1 + a/[n(a + 2)])^n - 1]^{1/n}.$$

Assuming, as before, that the bracket is equal to or greater than unity we have

$$x \geqq n(a^n + 2a^{n-1}).$$

Carrying this process through h steps we obtain

$$x \geqq n(a^n + ha^{n-1}), \tag{27}$$

provided

$$[(1 + a/[n(a + h)])^n - 1]^{1/n} \geqq 1 \tag{28}$$

for $h = 1, 2, 3, \cdots, h$.

Obviously this bracket is, for a fixed n, a decreasing function of h. Hence we need only determine the largest integral value of h for which the equality sign holds. This is equivalent to solving for h the equation

$$(1 + a/[n(a + h)])^n = 2.$$

We obtain

$$h = a(1/[n(2^{1/n} - 1)] - 1).$$

When $n = 3$ this gives us

$$h = 11a/39.$$

* One could, of course, iterate without this restriction but this results in a seemingly unprofitable form.

When $n=14000$, it gives

$$h = 34a/77 - ,$$

while

$$\lim_{n\to\infty} h = 34a/77 - .$$

(The difference in the two latter cases is beyond reasonable limits of computation.)

Hence for all values of n for which Fermat's Last Theorem has not been proved, namely those greater than 14000, we may take $h=34a/77$.

But our process of iteration permits only integral values for h. However, we surmount this difficulty by the following device. Continue the iterating process until we reach $[34a/77]$, which denotes the largest integer in $34a/77$. Call this K. We have then by relation (27)

$$x \geqq n(a^n + Ka^{n-1}).$$

Substituting the right member of this inequality for x in equation (22) gives

$$y \geqq n[a^n + (K+1)a^{n-1}].$$

But

$$K + 1 \geqq 34a/77.$$

Whence

$$y \geqq n[a^n + (34a/77)a^{n-1}].$$

Substituting the right member for y in relation (23) we obtain, by virtue of inequality (28),

$$\begin{aligned} x &\geqq n[a^n + (34/77)a^n] \\ &\geqq (111/77)na^n, \end{aligned} \tag{29}$$

which holds for all values of n greater than 14000.

Replacing a^n, $(\geqq z-y)$, by $(2cn^2+1)^n$ because of inequality (18), we have

$$x \geqq (111/77)n(2cn^2+1)^n, \qquad n \geqq 14000. \tag{30}$$

A similar procedure for $n=3$ gives

$$x \geqq (50/39)n(2cn^2+1)^n \geqq 26388, \quad \text{for} \quad n \geqq 3.$$

A CONTRIBUTION TO THE MATHEMATICAL THEORY OF BIG GAME HUNTING

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This little known mathematical discipline has not, of recent years, received in the literature the attention which, in our opinion, it deserves. In the present paper we present some algorithms which, it is hoped, may be of interest to other workers in the field. Neglecting the more obviously trivial methods, we shall confine our attention to those which involve significant applications of ideas familiar to mathematicians and physicists.

The present time is particularly fitting for the preparation of an account of the subject, since recent advances both in pure mathematics and in theoretical physics have made available powerful tools whose very existence was unsuspected by earlier investigators. At the same time, some of the more elegant classical methods acquire new significance in the light of modern discoveries. Like many other branches of knowledge to which mathematical techniques have been applied in recent years, the Mathematical Theory of Big Game Hunting has a singularly happy unifying effect on the most diverse branches of the exact sciences.

For the sake of simplicity of statement, we shall confine our attention to Lions (*Felis leo*) whose habitat is the Sahara Desert. The methods which we shall enumerate will easily be seen to be applicable, with obvious formal modifications, to other carnivores and to other portions of the globe. The paper is divided into three parts, which draw their material respectively from mathematics, theoretical physics, and experimental physics.

The author desires to acknowledge his indebtedness to the Trivial Club of St. John's College, Cambridge, England; to the M.I.T. chapter of the Society for Useless Research; to the F. o. P., of Princeton University; and to numerous individual contributors, known and unknown, conscious and unconscious.

1. Mathematical methods

1. THE HILBERT, OR AXIOMATIC, METHOD. We place a locked cage at a given point of the desert. We then introduce the following logical system.

AXIOM I. *The class of lions in the Sahara Desert is non-void.*

AXIOM II. *If there is a lion in the Sahara Desert, there is a lion in the cage.*

RULE OF PROCEDURE. *If p is a theorem, and "p implies q" is a theorem, then q is a theorem.*

THEOREM I. *There is a lion in the cage.*

2. THE METHOD OF INVERSIVE GEOMETRY. We place a *spherical* cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.

3. THE METHOD OF PROJECTIVE GEOMETRY. Without loss of generality, we may regard the Sahara Desert as a plane. Project the plane into a line, and then project the line into an interior point of the cage. The lion is projected into the same point.

4. THE BOLZANO-WEIERSTRASS METHOD. Bisect the desert by a line running N-S. The lion is either in the E portion or in the W portion; let us suppose him to be in the W portion. Bisect this portion by a line running E-W. The lion is either in the N portion or in the S portion; let us suppose him to be in the N portion. We continue this process indefinitely, constructing a sufficiently strong fence about the chosen portion at each step. The diameter of the chosen portions approaches zero, so that the lion is ultimately surrounded by a fence of arbitrarily small perimeter.

5. THE "MENGENTHEORETISCH" METHOD. We observe that the desert is a separable space. It therefore contains an enumerable dense set of points, from which can be extracted a sequence having the lion as limit. We then approach the lion stealthily along this sequence, bearing with us suitable equipment.

6. THE PEANO METHOD. Construct, by standard methods, a continuous curve passing through every point of the desert. It has been remarked* that it is possible to traverse such a curve in

* By Hilbert. See E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, 1927, vol. 1, pp. 456–457.

an arbitrarily short time. Armed with a spear, we traverse the curve in a time shorter than that in which a lion can move his own length.

7. A TOPOLOGICAL METHOD. We observe that a lion has at least the connectivity of the torus. We transport the desert into four-space. It is then possible* to carry out such a deformation that the lion can be returned to three-space in a knotted condition. He is then helpless.

8. THE CAUCHY, OR FUNCTIONTHEORETICAL, METHOD. We consider an analytic lion-valued function $f(z)$. Let ζ be the cage. Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}\,dz,$$

where C is the boundary of the desert; its value is $f(\zeta)$, *i.e.*, a lion in the cage.†

9. THE WIENER TAUBERIAN METHOD. We procure a tame lion, L_0, of class $L(-\infty, \infty)$, whose Fourier transform nowhere vanishes, and release it in the desert. L_0 then converges to our cage. By Wiener's General Tauberian Theorem,‡ any other lion, L (say), will then converge to the same cage. Alternatively, we can approximate arbitrarily closely to L by translating L_0 about the desert.§

2. Methods from theoretical physics

10. THE DIRAC METHOD. We observe that wild lions are, *ipso facto*, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.

11. THE SCHRÖDINGER METHOD. At any given moment there is a positive probability that there is a lion in the cage. Sit down and wait.

12. THE METHOD OF NUCLEAR PHYSICS. Place a tame lion in the cage, and apply a Majorana exchange operator|| between it and a wild lion.

As a variant, let us suppose, to fix ideas, that we require a male lion. We place a tame lioness in the cage, and apply a Heisenberg exchange operator¶ which exchanges the spins.

13. A RELATIVISTIC METHOD. We distribute about the desert lion bait containing large portions of the Companion of Sirius. When enough bait has been taken, we project a beam of light across the desert. This will bend right round the lion, who will then become so dizzy that he can be approached with impunity.

3. Methods from experimental physics

14. THE THERMODYNAMICAL METHOD. We construct a semi-permeable membrane, permeable to everything except lions, and sweep it across the desert.

15. THE ATOM-SPLITTING METHOD. We irradiate the desert with slow neutrons. The lion becomes radioactive, and a process of disintegration sets in. When the decay has proceeded sufficiently far, he will become incapable of showing fight.

16. THE MAGNETO-OPTICAL METHOD. We plant a large lenticular bed of catnip (*Nepeta cataria*), whose axis lies along the direction of the horizontal component of the earth's magnetic field, and place a cage at one of its foci. We distribute over the desert large quantities of magnetized spinach (*Spinacia oleracea*), which, as is well known, has a high ferric content. The spinach is eaten by the herbivorous denizens of the desert, which are in turn eaten by lions. The lions are then oriented parallel to the earth's magnetic field, and the resulting beam of lions is focussed by the catnip upon the cage.

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* H. Seifert and W. Threlfall, Lehrbuch der Topologie, 1934, pp. 2–3.

† *N.B.* By Picard's Theorem (W. F. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 1928, p. 748), we can catch every lion with at most one exception.

‡ N. Wiener, The Fourier Integral and Certain of its Applications, 1933, pp. 73–74.

§ N. Wiener, *l. c.*, p. 89.

|| See, for example, H. A. Bethe and R. F. Bacher, Reviews of Modern Physics, vol. 8, 1936, pp. 82–229; especially pp. 106–107.

¶ *Ibid.*

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

THE FUNCTION CONCEPT IN ELEMENTARY TEACHING AND IN ADVANCED MATHEMATICS*

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The ways in which the function concept occurs throughout mathematics have been dwelt upon by many writers. One of the first to emphasize its vital importance in all mathematical teaching was Felix Klein, who played such a prominent rôle in mathematics a generation ago in Germany. In this country, the Report of the National Committee on the Reorganization of Secondary-School Mathematics brought before teachers of mathematics the fundamental importance of the function concept and of functional thinking. Through this Report, and in other papers, I myself had a part in stressing the importance of these ideas.

In its elementary phases, the function concept agrees to all intents with the idea of the interdependence of quantities, though the more refined definitions of function do not *require* that there be dependence. Broadly speaking, then, the function concept in elementary work means the dependence of one quantity upon one or more others. The simple questions that arise are two-fold: (1) Upon what quantities does one in which we are interested depend? and (2) What is the precise manner in which the one depends upon the other or others? Both of these questions are normally present in practically every mathematical subject; they occur also very frequently in problems of living, and in the several sciences. Properly to be within the range of functional thinking, we need not concern ourselves necessarily with *both* of these questions: If either of the two questions is discussed, functional thinking is being done. Thus, if we think of the resistance of the air to the motion of an automobile, functional thought is involved when we say that the resistance depends upon the speed. To think functionally, it is not necessary to know the precise formula; indeed, I doubt very much whether any such formula can be given by anyone. Any observation, such as that an increase in speed causes an increase in resistance, falls under the head of functional thought. The search for experimental data is another phase of functional thinking, through which statistical information regarding

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resistances at various speeds is obtained. The drawing of a graph gives still more definite information about the nature of the relationship.

That the bare fact of dependence requires intelligent thought may be illustrated best by an example. For many centuries, the engineers of the Roman Empire constructed aqueducts, and they supplied water to cities for domestic purposes and to farmers for irrigation. The quantities of water furnished were determined solely by the size of the opening through which the water was delivered. Apparently the first man to notice that the amount of water that would pass through a given opening depended also on the *pressure* (or "head") was Leonardo da Vinci. He also stated the approximate formula that is still in use by engineers. In this country, even during the latter half of the nineteenth century, water rights for irrigation were stated in "inches," that is, per square inch of opening, without regard to the pressure; many legal battles have resulted, especially in California, as a result of this practice. I have stated this as a conspicuous instance of the failure, even by fairly intelligent persons vitally concerned, to note the dependence of one quantity upon another. We can not expect, then, that even fairly intelligent people will notice upon what quantities a given one depends, though they themselves may be deeply interested. Most certainly we can not expect children to know without training anything whatever about the ways in which quantities of very simple kinds are related.

In this paper, it is not my purpose to enter into detail regarding any one subject or any one school-age, more particularly because studies abound regarding such isolated instances; rather my purpose will be to show the persistence of the function idea through every stage of mathematical development. Finally, I shall touch upon the advanced studies which center around the function concept.

The emphasis placed upon functional thinking as the great central theme of all mathematics in the Report of the Committee mentioned above, was entirely sound and was very timely. The criticisms of mathematics as a school subject have been very severe and very bitter during the last twenty years. It has been said by many educators that mathematics has no relationship to life or to the affairs of the world. Harsh criticism has been leveled against the formalisms of the traditional courses in algebra and geometry in the schools; and these criticisms have had a considerable basis of truth. School authorities have insisted upon what is called "integration" of school subjects, which means in mathematics the constant attention to questions that arise in physics and in other sciences, and in the outside world. Often these educational leaders have concluded, from a superficial examination of traditional courses in mathematics, that mathematics does not lend itself to integration with physics, with the other sciences, or with affairs of life. If you will examine again the traditional courses, even in textbooks now in use, you will see that they are not greatly to be blamed for such a false conclusion. I do not blame them severely, though I do hold that expert advice on school subjects should have been based upon something more than a superficial examination of what has been done in the past.

Something more than a superficial examination will make clear that quantities and relationships between quantities do occur—in fact they abound—in physics and in all science. They do occur—in fact they abound—in the life of the world. Can we then integrate mathematical teaching with life and with science? We can indeed if we choose to give more than lip-service to the Report of the Committee mentioned above, and if we stress the function concept as the central theme in every course. Moreover, we shall not lack for examples and for illustrations if we do so. I propose to run through the whole gamut of the elementary mathematical subjects with you, to see where we may well alter the traditional presentations.

Even in elementary arithmetic, functional thinking may be carried on at every stage. The *word* "function" should not be used, of course, but the *word* is not the important thing. If one apple costs two cents, how much will five cost? Every such question in arithmetic associates two or more related quantities. The questions are childish, of course; they should be. What is lacking is appreciation of what quantities affect the answers. We should ask: "What do you need to know?" Why, of course, the price of one apple and the number of apples. In such childish ways the idea of relationship between quantities is born in the mind. We may keep it fresh, or we may let it die. Try to think, if you will, of some topic in arithmetic that does not involve related quantities. There are none. What is lacking is our own appreciation, and our insistence that the child see what the relationship is, what quantities are tied together, what must be known to find a desired answer. I can not detail them all. I defy you, however, to find any topic in arithmetic in which this question is not the paramount one. It is often buried beyond recall in a maze of formalisms.

The story in algebra is not far different. I am aware, of course, that traditional courses emphasize formalisms that are far from functional thinking. When they do, I challenge you to discover any thought whatever in what is being done. Thus overemphasis has been placed upon factoring; you will find that I do not go beyond the Report mentioned above when I condemn unreservedly any elaborate exercises in factoring. Solution of equations is made to appear the central theme in many courses in algebra, and the old-time texts went to absurd extremes in intricacies of simultaneous quadratics that are quite useless. What is really true is that the setting up of any equation does bring in functional thought. There must be some unknown quantity; its value is sought *always* by giving some other quantity or quantities which are related to the unknown one. We set up the equation *always* by expressing the connection that exists between the known and the unknown quantities. This is the prime purpose of the entire process. Merely to know how to solve the equation after it has been set up gives no fundamental power over real situations. This is emphatically proved by the inability of students to solve problems stated in English. They often say "If you will show me how to set up the equation, I can solve it." The truth is that the ability to recognize the relations between

the quantities mentioned in the problem is the vital thing, and the mechanical solution of the resulting equation is a minor matter.

In geometry, as it is traditionally taught, the principal functional thinking is in the mensuration formulas. That the area of a square is the square of the side is a typical instance. Other areas (of triangles, rectangles, circles, etc.) and volumes of solids are given by specific formulas. Often, however, little stress is placed upon real comprehension: thus that a circle twice as wide as another has four times the area, is often regarded as unimportant. But there are many other functional ideas in geometry, most of which are suppressed. Thus if a triangle is determined by two sides and the included angle, the other sides *must be* functions of the three given parts. The idea of trigonometric calculation is intrinsic in the congruence theorems. In fact, the better texts, both in geometry and in algebra, now give the elements of trigonometric calculation for right triangles. Throughout the geometry, in a similar manner, the question of dependence of parts of a figure upon other parts can be emphasized, and it is often far more important than some of the alleged logic that is more often too much emphasized. For example, in the relationships between arcs and angles the relationships which are really vital are too often made secondary, and such fallacious logic as that of the old treatments of incommensurables is called important. The result is that our strenuous efforts to give "mental discipline" miscarries, for there is real discipline to be had here in training the student to think functionally. It is not incidental, but quite vital, throughout geometry, that the student be trained to see the relationships between the various parts of each figure. This is, in fact, the real *geometry*.

Trigonometry is functional on its face. It is the first place in which we traditionally use the word "function." So be it. I have no brief for introduction of mere words into our teaching. Even in trigonometry, however, there has been too great a tendency to emphasize dreary and profitless formalisms that involve no real thought at all, as in the numerous problems in so-called trigonometric identities. Except for a very few simple ones, these might better be sacrificed to more thought about the way in which the trigonometric functions vary. Thus the graphs of the trigonometric functions, and some thought about quantities in physics and elsewhere which vary trigonometrically, is far more vital.

College mathematics has been discussed less than has secondary mathematics. The same questions arise, however, and the traditional courses dwell all too little upon functional thinking. In analytic geometry, for example, what is of primary importance is to know how to indicate graphically the behavior of relations which are given by equations. By contrast, it is not particularly important to discuss at length the general equation of the second degree, and the general conic. Far simpler questions that are often omitted entirely are to picture graphically the relations between quantities connected by such simple equations as $xy=k$, (inverse variation); $y=x^n$ for a variety of values of n;

$y=(ax+b)/(cx+d)$; $y=e^x$; $y=\sin x$; *etc.* If these are accompanied by "integrating" explanation of instances of such relations in science, great insight may be given into a variety of functional thinking. I should not omit to mention the manner of dealing with equations of tangents to curves. Here I may mention the traditional process of finding (by special devices) the equations of tangents to ellipses and hyperbolas, to the exclusion even of the tangent to the curve $y=x^n$ for integral values of n. These special devices leave the student wholly unaware that he is dealing with the rate of increase of a function. By comparison, the finding of tangents to ellipses and hyperbolas, which is really far more intricate, should be postponed to the calculus, and the far simpler cases of explicit functions, particularly the simple polynomial functions, should receive the great emphasis. Throughout the whole course, it is easy to see in any text whether or not the author himself realizes the importance of the function concept. In so far as he does, and in so far as the teacher emphasizes the functional thinking that is involved, the course will increase in value to the student, either in mathematics itself or in any of its applications.

Similar differences exist throughout the calculus. If the derivative is presented as a formalized thing, the rules can be learned and numerous examples can be handed in, without the slightest comprehension on the part of the student of the real meaning of the calculus. I know, because I did it just that way. The calculus begins to have vivid meaning when the student realizes that the derivative of a function represents the rate of increase. This should be held before his attention at all times. Thus the finding of equations of tangents to curves is not particularly important in itself; the important thing is the finding of the *slope* of the tangent, and the realization of its significance. It is easy to tell in any class or in any textbook whether or not the attention is upon functional thinking or upon the formal solution of set exercises. If the examples that are given are frequently or principally the familiar ellipses and hyperbolas, and such curves as $x^{2/3}+y^{2/3}=a^{2/3}$, you may be fairly sure that not much functional thinking is being done. If, on the other hand, there is quite a good proportion of examples that are based on $y=x^n$, $y=$ a polynomial, $y=e^x$, $y=\sin x$, and so on, there is a higher probability that the student is getting some real comprehension. If there is much talk of geometrical properties of conjugate diameters and poles and polars, there is not much chance that the student is going to have any use for what he is learning; I know, for that is what I did. If there is a good deal of talk about rates of increase, and of speed and acceleration, and of relative rates, then there is more likelihood that functional thinking is going on. It does not take great practice to know whether or not the textbook, or the teacher, is in sympathy, or out of sympathy, with functional thinking. These things, both their lack and their presence, characterize every chapter of every text, and every lesson of every class.

I may call "advanced" the mathematics beyond the calculus. In this field, through the junior and senior years and in the graduate schools, mathematics normally divides into several branches. Time was when a student might pro-

ceed even to the doctorate without much more attention to the function concept than that which he got in his course in calculus, and his course in calculus may have been extremely weak in functional thinking, as was my own. I learned mainly the properties of many intricate curves, and much about such geometric entities as polars and subtangents. I did not know (though I was given high grades) that a derivative could mean a rate of increase, nor that an integral could be thought of as a limit of a sum. I knew much geometric lore, most of which I found to be useless, but I knew almost no calculus, that is, I knew nearly nothing about functional thought.

We still have, and we should have, specialists in geometry. No longer, however, can anyone be regarded as a mathematical graduate, or graduate student, if he knows only geometric things, and does not know functional thinking. The day of over-emphasis upon pure geometry has passed. Professor Julian Coolidge read before the Mathematical Association an excellent lecture in which he pointed out that the day of geometrical training, to the exclusion of all else, has passed. I believe that he—and perhaps some of you—acknowledged that with regret. It does not mean that any part of geometry is dead; that will never happen. It means, however, that fundamental training in functional thinking is insisted upon now, and will be throughout the future, for all who specialize in mathematics. In this, there is no cause for regret.

Courses in advanced calculus, or (what is the same thing) in mathematical analysis, are now part of every course for every student who seeks a degree in mathematics. In such a course, without too much display, the primary consideration is a more thorough training in functions and in functional thinking. Here for the first time an accurate idea of "function" becomes necessary. We say now, for the first time, that one quantity is a function of another if, when any value of the second is given (in a specified range), the value of the first one is determined. We proceed to examine more carefully all of the ideas of the calculus, such as limit, continuous function, derivative, integral, and the various theorems. We discuss more accurately the number system itself, and the associated ideas of sets of numbers. We deal with extensions of the elementary functional concepts, such as line-integrals, infinite series, and the like. In all, such a course should increase very greatly the student's comprehension of functional behavior and his power to think functionally. With such a course every teacher of a more elementary course becomes function-conscious, and he will normally give his own students more power to think functionally. Without such a course, in my opinion, no teacher should be permitted to teach any collegiate subject in mathematics.

For a moment, I would like to call your attention to some traditional aspects of the more advanced courses that bear the name "function." These are, specifically, the courses (usually graduate courses) entitled *Functions of a real variable*, and *Functions of a complex variable*. Studies in these two fields have been very intensive and the literature is now so formidable that I do not believe that any one man is master of it all. In the courses, naturally, only the beginning, and

only the fundamental ideas, are considered. I shall not attempt to outline even these first courses: either you have had them, in which case my explanations would be superfluous, or else a quick summary would be hard to grasp. What I do want to try to do is to contrast the very wide difference in attitude which characterizes the traditional courses in these subjects. When we speak of functions of a real variable, we allow ourselves every latitude: the definition of *function* is perhaps even more refined than that which I gave above, and no functions are excluded from discussions, except by hypothesis in each theorem. Thus we do not assume that a function must be continuous, even at any one of the points of its graph. Certainly we do not assume that the function has a derivative. We may discuss, to be sure, certain classes of functions: thus we may and we do derive theorems on the properties of continuous functions, or on functions of limited variation, or on functions that possess derivatives. When we do so, however, we recognize openly that we are segregating out a particular class of functions; nobody is deceived into thinking that *all* functions possess the properties that are enunciated for the functions of a particular class.

In the traditional course on functions of a complex variable, however, a very different procedure is traditional. It appears to me that it deceives many students. By tradition, we assume at the start that the function (of a complex variable) has a derivative, at least except for certain few values of the independent variable. Having done so, we tend traditionally to lose sight of the possible existence of other functions. This is all the more serious because the restriction imposed by the hypothesis that the derivative exists is far and away a more strenuous restriction than would be the similar assumption for functions of real variables. I myself was perplexed by this difference in traditional attitude, and I have been among those who have tried in recent years to extend to a base as broad as that for real variables the theory of functions for complex variables. Considerable progress has been made in this field, and I am quite sure that the future will see the treatments in the two fields placed openly upon the same basis. To do so will remove many of the perplexities which beset the beginning student. This will not prevent a treatment of functions that possess derivatives, as it does not prevent such a treatment now in real variables, but the student will not be deceived into thinking that the theorems derived are true for *all* functions.

Thus in the traditional theory for complex variables, *every* function (single-valued) is expansible in Taylor Series. It was not clear to me when I learned this that this theorem is true *only* for the restricted group of functions that possess derivatives. Hence the obvious fact that functions of real variables do not obey the same theorem seemed quite weird to me. I believe that this one case is responsible for many of the quite false statements that are made in texts on the calculus about expansion of functions in Taylor series.

Many other such apparent discrepancies exist between the two theories of functions, the one of real variables, and the other of complex variables. Thus a

function of a complex variable (in the restricted sense) is completely determined by its values in any small two-dimensional region; the apparently analogous theorem is wholly false for real variables. I shall not extend this discussion, though very numerous instances exist, some of which may have confused and mystified some of you, as they did me. Such confusion disappears entirely when we recognize openly that the traditional theory of functions of a complex variable is valid only for a special class of those functions. I hope that this bare statement may be of assistance to many of you who may have had difficulties similar to my own.

In closing, I may point out also that there are many theorems in the traditional theory of functions of a complex variable which happen to be valid for very broad classes of functions, so that the restriction traditionally made is not always needed. Thus the theorem that the absolute value of a single-valued function of a complex variable can not have a maximum happens to be true for very much wider classes of functions than those of the traditional theory. It is not necessary to assume that the derivative of the function exists; indeed the hypothesis that the ordinary jacobian exists and is different from zero, is sufficient. The more complicated theorems, usually called the Liouville theorems, also hold good under such a hypothesis. An interesting question is to discover how much the hypotheses of the traditional theorems may be reduced without affecting the validity of the conclusions.

I will not strain your patience further. My intention has been to review briefly the entry of the function concept into all mathematics, from the first steps of the arithmetic of children to the more advanced conscious treatment of the theory of functions as such. I trust that you will agree—as the Report of the Committee has it—that this is the one theme which tends to unify all of mathematics and to permit its integration with life and with science. Would that all teachers of mathematics, all students of mathematics, knew this truth! Would that all the professed leaders of education were aware of it!

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON CERTAIN FORMULAS USED IN SAMPLING THEORY

J. F. KENNEY, Northwestern University

In the October 1937 number of this MONTHLY, p. 491, C. N. Mills is reported to have given the formula*

$$\frac{1}{N}\left\{\sum (b-a)^2 - \frac{\sum (b-a)^2}{N}\right\}^{1/2}$$

as being better adapted to numerical computation than the usual formula for the standard error of the difference between the means of paired items a_i, b_i $(i=1, 2, \cdots, N)$. The formula printed is not correct. It should be

$$\frac{1}{N}\left\{\sum (b-a)^2 - \frac{(\sum b - \sum a)^2}{N}\right\}^{1/2},$$

and the derivation involves the assumption that N is large.

This suggests an instance where it seems that an incorrect, or inappropriate, formula is given in certain textbooks on sampling theory. Let $\bar{x}_1$, $\bar{x}_2$ be the means and s_1, s_2 the standard deviations of two independent samples consisting of N_1 and N_2 variates, respectively, from a normal universe with mean m and variance σ^2. Then the variance of the difference between the means is equal to $\sigma^2(N_1+N_2)/N_1N_2$, and, as is well known [**1**; p. 357], the variable

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sigma}\left(\frac{N_1N_2}{N_1+N_2}\right)^{1/2}$$

is normally distributed with mean zero and unit standard deviation. However, in practical problems σ^2 is seldom available and must be estimated from the samples. Let $\dot{\theta}$ be a function of the variates supplied by a sample, or samples, for estimating a population parameter θ. If the expected value of $\dot{\theta}$ is θ then $\dot{\theta}$ is called an unbiased estimate of θ. An unbiased estimate of σ^2 is

$$\dot{\sigma}^2 = \frac{N_1s_1^2 + N_2s_2^2}{N_1+N_2-2}.$$

Fisher [**2**] and others [**1** and **3**] have shown that the variable

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\dot{\sigma}}\left(\frac{N_1N_2}{N_1+N_2}\right)^{1/2} \tag{1}$$

* *Note by Editor.* The formula submitted by Professor Mills was incorrectly printed. The numerator of the fraction was $[\sum(b-a)]^2$, and not $\sum(b-a)^2$. J.R.M.

is distributed in accord with the "Student" curve

$$F_n(t) = K_n(1 + t^2/n)^{-(n+1)/2}, \tag{2}$$

where $1/K_n = n^{1/2}B(n/2, 1/2)$, B being the Beta function, and $n = N_1 + N_2 - 2$ is the number of so-called "degrees of freedom."

Now as N_1 and N_2 become large, $(N_1+N_2)/(N_1+N_2-2)$ tends toward unity and (1) tends toward the value

$$t = \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2/N_2 + s_2^2/N_1)^{1/2}} \cdot \tag{3}$$

Since (1) is asymptotically normally distributed, as can be seen from (2), the procedure of referring (3) to a normal probability scale would not be invalid to any appreciable extent for large values of N_1 and N_2. But the procedure commonly given in textbooks for testing a null hypothesis [4] that two samples are from the same universe, using their means as a criterion of judgment, is to refer

$$t = \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2/N_1 + s_2^2/N_2)^{1/2}}$$

instead of (3) to a normal probability scale.

It is interesting to observe that if one of the samples, say N_2, becomes infinitely large so that $\bar{x}_2$ becomes m and s_2 becomes σ, then (3) becomes

$$t = \frac{(\bar{x}_1 - m)(N_1)^{1/2}}{\sigma} \tag{4}$$

which, if the subscripts are dropped, is the formula used in testing a null hypothesis that a given sample comes from a universe with a proposed mean. If the universe is normal then (4) is normally distributed. When σ is unknown we may use in its place an estimate, $\dot{\sigma}$, defined by

$$\dot{\sigma}^2 = s^2N/(N - 1),$$

where s^2 is the variance of the sample. But if we substitute $\dot{\sigma}$ for σ in (4) and calculate

$$t = \frac{(\bar{x} - m)(N)^{1/2}}{\dot{\sigma}} = \frac{(\bar{x} - m)(N - 1)^{1/2}}{s}, \tag{5}$$

we are not justified in asserting that (5) is normally distributed unless N is large. As a matter of fact, (5) is distributed in accord with (2) for $n = N - 1$.

In 1925, "Student" published in Metron [5] an extensive table of the probability integral of (2). More recently, Fisher [6] has given a short table of the probability $P_n(t)$ for deviations outside $\pm t$, for values of t and n commonly met in practice, where

$$P_n(t) = 1 - 2\int_0^t F_n(t)dt. \tag{6}$$

If a sample, or samples, could not reasonably have arisen from the universe proposed, the null hypothesis is said to be refuted for the level of significance adopted. If the significance test yields a verdict of "not significant" at the probability level adopted, we say that the null hypothesis is not refuted or contradicted at that level. It is open to the investigator to be more or less exacting concerning the smallness of the probability he would require before he would be willing to admit that his test has demonstrated a significant result. However, the following rule is conventional among certain workers. If $P_n(t) \geqq .05$, t is not significant; if $P_n(t) \leqq .01$, t is significant; if $.05 > P_n(t) > .01$, any conclusion about t is doubtful until more information is available. The same rule is used when it is valid to replace $F_n(t)$ by the normal probability function in (6). Others may prefer a more conservative level of significance.

References

1. Dunham Jackson, The Theory of Small Samples, this MONTHLY, vol. 42, 1935, pp. 344–364.

2. R. A. Fisher, Applications of "Student's" Distribution, Metron, vol. 5, 1925, no. 3, pp. 90–104.

3. P. R. Rider, A Survey of the Theory of Small Samples, Annals of Mathematics, (2), vol. 31, 1930, pp. 577–628.

4. R. A. Fisher, The Design of Experiments. Oliver and Boyd, London.

5. "Student," Metron, vol. 5, 1925, no. 3, pp. 114–120. "Student's" real name was W. S. Gosset. An interesting biographical sketch of Mr. Gosset has recently been given by J. Neyman in Journal of American Statistical Association, vol. 33, pp. 226–228.

6. R. A. Fisher, Statistical Methods for Research Workers. Oliver and Boyd, London.

A NOTE ON HEDGING

H. D. LARSEN, University of New Mexico

A problem of considerable interest to elementary students and to others concerns the conditions under which a gambler can so hedge his bets that a profit is assured. Consider a contest in which there are three entries, A, B, and C, one of whom must win. Let their respective odds of winning be *quoted* as a to b, c to d, and e to f. Thus the contestants' "probabilities" of winning may be defined as

$$P_A = a/(a+b), \qquad P_B = c/(c+d), \qquad P_C = e/(e+f).$$

Suppose that we have a sum to wager on this contest, and that we distribute it over the three entries in the following proportion: x on A to win, y on B to win, and z on C to win, where $x+y+z=1$. Then if A wins, our gain (profit or loss) is

$$G_A = \frac{b}{a}x - (y+z) = \frac{b}{a}x - (1-x)$$

$$= x/P_A - 1.$$

Similarly,

$$G_B = y/P_B - 1,$$

and

$$G_C = z/P_C - 1.$$

Let us now see if we can so place our respective bets that the same profit K ensues no matter which entry wins. We wish that

$$G_A = G_B = G_C = K,$$

whence

$$x/P_A = y/P_B = z/P_C = K + 1, \tag{1}$$

i.e.,

$$x:y:z = P_A:P_B:P_C. \tag{2}$$

Thus, if we distribute our bets in proportion to the "probabilities" of winning we are assured of a gain K. In order that our gain be an actual profit and not a loss, a further condition is necessary. By (1) we have

$$x = (K + 1)P_A, \qquad y = (K + 1)P_B, \qquad z = (K + 1)P_C,$$

and, adding,

$$1 = (K + 1)(P_A + P_B + P_C),$$

whence

$$K = 1/(P_A + P_B + P_C) - 1,$$

which is positive if

$$P_A + P_B + P_C < 1. \tag{3}$$

An examination of (2) and (3) will show that they are both necessary and sufficient conditions for a constant profit K. These results can be generalized in an obvious manner. If (3) is not satisfied, but $P_A+P_B+P_C>1$, we need only bet *against* each entry to be assured of a profit. If $P_A+P_B+P_C=1$, it is not possible to hedge the bets in such a way that a profit is certain.

An interesting application of the above theory is found in the case of a contest in which there are but two entries. The following news item appeared last fall: "The betting odds listed the Tigers as a one to four choice $\cdots$. For those who liked the Bronchos, bookmakers were laying eleven to five." Thus the odds quoted for this game were (a) 11 to 5 that the Tigers win, 1 to 4 that the Bronchos win; or (b) 5 to 11 that the Bronchos win, 4 to 1 that the Tigers win. The condition (3) is satisfied by (a) only, in which case

$$P_A = \frac{11}{16}, \qquad P_B = \frac{1}{5}, \qquad P_A + P_B = \frac{71}{80} < 1.$$

To obtain the same profit no matter which team wins the game, we should distribute our bets by (2) in the proportion 11/16:1/5, or 55:16. Suppose we place \$550 on the Tigers at 11 to 5, and \$160 on the Bronchos at 1 to 4. Then if the Tigers win, we gain 5/11 of \$550, or \$250, and lose \$160. If the Bronchos win, we lose \$550 but gain 4×\$160, or \$640. In either case, our net profit is \$90.

APPROXIMATE EVALUATION OF CERTAIN ELLIPTIC INTEGRALS

E. C. Kennedy, The Rice Institute

Consider the Elliptic Integral of the second kind:

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^\phi \sqrt{\cos^2 \phi + h \sin^2 \phi}\, d\phi,$$

where $1-k^2=h$. On expanding and integrating we get

$$E(k, \phi) = (1 - h/2) \sin \phi + \tfrac{1}{2}h \log \tan (45° + \phi/2)$$
$$- \frac{h^2}{8}\int_0^\phi \tan^3 \phi \sin \phi\, d\phi + \cdots .$$

Thus we obtain the following simple approximation formula which holds for small values of ϕ and values of k^2 near one:

$$E(k, \phi) = (1 - h/2) \sin \phi + \tfrac{1}{2}h[\log \tan (45° + \phi/2)].$$

As an illustration of its applicability consider

Example 1. Given $\phi=40°$, $k^2=.8$. Find E.

$$E(k, \phi) = .9 \sin 40° + .1 \log \tan 65°.$$
$$E(k, \phi) = .5785 + .0763 = .6548 \text{ (Tables give .6551)}.$$

Example 2. Given $\phi=22°36'$, $k^2=.92$. Find E.

$$E(k, \phi) = .96 \sin 22°36' - .04 \log \tan 56°18'.$$
$$E(k, \phi) = .368923 + .016204 = .38513 \text{ (Tables give .3851)}.$$

In the latter example our formula probably gives the result quicker than does the tables. It would be impracticable to use formula 527, Peirce's Tables, on such a problem. The beauty of our formula lies in the fact that we can readily estimate the magnitude of our error and there is no increased labor involved when ϕ is not an even degree.

A correspondingly simple formula may be obtained for

$$F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Thus

$$F(k, \phi) = (1 + h/4) \log \tan (45° + \phi/2) - h/4 \tan \phi \sec \phi + \cdots,$$

where $h=1-k^2$, ϕ is small, and k^2 is near one.

Example 3. Given $\phi=30°$, $k^2=.88$. Find F.

$$F(k, \phi) = (1.03) \log \tan 60° - .03 \tan 30° \sec 30°.$$

$$F(k, \phi) = .5658 - .0200 = .5458 \text{ (Tables give } .5456).$$

ON THE AVERAGE-SLOPE METHOD OF SOLVING DIFFERENTIAL EQUATIONS

A. W. RANKIN, General Electric Company

In the solution by successive approximations of differential equations of the first order which may be solved for y', a method sometimes employed is the following:

Given

$$(1) \qquad y' = dy/dx = f(x, y),$$

we wish a solution which passes through the point (x_0, y_0). If y_0' is the slope at the point (x_0, y_0), a first approximation to the value of y_1 is

$$(2) \qquad y_1 = y_0 + \Delta y_0',$$

where Δ is the increment of x. Substituting the value of y_1 from (2) in (1), together with the value $x_1=x_0+\Delta$, gives a first approximation to the true slope at (x_1, y_1). A second approximation to the true value of y_1 is then obtained by using in (2) the average of y_0' and the first approximation to y_1'. This second approximation to y_1, when substituted in (1), gives a second approximation to the slope at the point (x_1, y_1); and a third approximation to y_1 is obtained by using the average of y_0' and the second approximation to y_1', substituted again in (2). This method is continued until two successive values of y_1 differ by a sufficiently small amount. The simple example shown below, however, proves that, even though it be employed indefinitely, this method does not necessarily lead to the correct value of y_1'.

Consider the equation:

$$(3) \qquad y' = y.$$

We shall seek the solution passing through the point (x_0, y_0). We have

$$(4) \qquad y_0' = y_0,$$

and writing Δ for Δx

$$(5) \qquad y_{1,1} = y_0 + \Delta y_0,$$

where the subscript (1, 1) means the first approximation to the value of y_1. Likewise, the subscript (1, 2) shall mean the second approximation to the value of y_1, *etc.*; and Δ is the increment of x.

Substituting (5) in (3), we obtain

(6) $$y'_{1,1} = y_0 + \Delta y_0.$$

The average of (4) and (6) is

(7) $$y_0 + y_0\Delta/2.$$

Substituting the average slope from (7) back into (5), we have

(8) $$y_{1,2} = y_0 + y_0\Delta + y_0\Delta^2/2.$$

Substituting (8) in (3), we obtain

(9) $$y'_{1,2} = y_0 + y_0\Delta + y_0\Delta^2/2.$$

The average of (4) and (9) is

(10) $$y_0 + y_0\Delta/2 + y_0\Delta^2/4.$$

Substituting the average slope given by (10) back into (5), we get

(11) $$y_{1,3} = y_0 + y_0\Delta + y_0\Delta^2/2 + y_0\Delta^3/4.$$

The continuation of this method will give as the nth approximation to the value of y_1:

(12) $$y_{1,n} = y_0(1 + \Delta + \Delta^2/2 + \Delta^3/3 + \cdots + \Delta^n/2^{(n-1)}).$$

Letting n approach infinity in (12), we have

(13) $$\lim_{n\to\infty} y_{1,n} = y_0 \frac{1 + \Delta/2}{1 - \Delta/2};$$

(13) is then the value of y_1 obtained through the average-slope method described above. However, integration of (3) and substitution of the given point (x_0, y_0) gives $y = y_0e^{(x-x_0)}$. Substituting $x_1 = x_0 + \Delta$, we get $y_1 = y_0e^{\Delta}$.

CONCERNING NAPIER'S RULES, A CORRECTION

W. R. Ransom, Tufts College

In a note in the January Monthly, I remarked that Napier's own proof seems not to have been reproduced in any American textbook. It has now been called to my attention that it does appear in R. E. Moritz's *Plane and Spherical Trigonometry* (John Wiley and Sons, New York, 1913) on page 18 of the part on spherical trigonometry. I regret that I did not have this book at hand when the previous note was written.

NOTE CONCERNING PROFESSOR DODD'S PAPER

Professor E. L. Dodd has reported that after it was too late to make a change in his recent article (this Monthly May, 1938) he discovered that the results given in his Sections 2 and 3, pp. 302–304, had been published by Professor Dunham Jackson in: "Note on quartiles and allied measures," *Bulletin of the American Mathematical Society*, vol. 29, 1923, pp. 17–20. E. J. M.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

A Text-Book of Convergence. By W. L. Ferrar. London and New York, Oxford University Press, 1938. 7+192 pages. $3.50.

Portraits of Eminent Mathematicians. By D. E. Smith. Portfolio II. New York, Scripta Mathematica, 1938. $3.00.

Principles of Mathematics. By B. Russell. Second Edition. New York, W. W. Norton and Company, Inc., 1938. 39+534 pages. $5.00.

Duodecimal Arithmetic. By G. S. Terry. New York, Longmans, Green and Company, 1938. 407 pages. $7.50.

Magic Squares of $(2n+1)^2$ *Cells.* By M. J. van Driel. London, Rider and Company, 1936. 90 pages. $2.75.

Trigonometry. By A. R. Crathorne and E. B. Lytle. Revised Edition. New York, Henry Holt and Company, 1938. 9+191 pages. Logarithmic and Trigonometric Tables. 16+95 pages. $2.00 with tables. $1.70 without tables.

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur. Revised Edition. New York, Farrar and Rinehart, Inc., 1938. 20+584 pages. $3.50.

Functions of Real Variables. By W. F. Osgood. Reprint of 1936 edition published in Peking. New York, G. E. Stechert and Company, 1938. 399 pages. $4.00.

Functions of a Complex Variable. By W. F. Osgood. Reprint of 1936 edition published in Peking. New York, G. E. Stechert and Company, 1938. 257 pages. $3.00.

A Course of Pure Mathematics. By G. H. Hardy. Seventh Edition. Cambridge, University Press, and New York, The Macmillan Company, 1938. 12+498 pages. $3.75.

Essentials of Engineering Mathematics. By J. P. Ballantine. New York, Prentice-Hall, Inc., 1938. 11+502+76 pages. $3.75.

Calculus. By E. S. Smith, M. Salkover and H. K. Justice. New York, John Wiley and Sons, 1938. 12+558 pages. $3.25.

Construction, Classification and Census of Magic Squares of Order Five. By A. L. Candy. Lincoln, Nebraska, A. L. Candy, 1938. $1.00.

Aspects of Science. By T. Dantzig. New York, The Macmillan Company, 1938. $3.00.

The Queen of the Sciences. By E. T. Bell. New York, G. E. Stechert and Company, 1938. 138 pages. Reprint. $1.50.

An Introduction to Business Statistics. By J. R. Stockton. New York, D. C. Heath and Company, 1938. 5+378 pages. $3.00.

Erbmathematik. Theorie der Vererbung in Bevölkerung und Sippe. By H. Geppert and S. Koller. Leipzig, Quelle and Meyer, 1938. 8+228 pages. RM 16.

Actualités Scientifiques et Industrielles. Paris, Herman et Cie.

502. *Nécessaire mathématique.* By M. Curie and M. Prost. 1937. 20 fr.

542. VII. *Recherches sur la théorie cinétique des liquides (première partie): Fluctuations en densité.* By J. Yvon. Théories Mécaniques (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 63 pages. 18 fr.

543. VIII. *Recherches sur la théorie cinétique des liquides (deuxième partie): La propagation et la diffusion de la lumière.* By J. Yvon. Théories Mécaniques. (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 133 pages. 18 fr.

544. IX. *Les Phénomènes d'auto-oscillation dans les installations hydrauliques.* By Y. Rocard. Théories Mécaniques. (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 69 pages. 18 fr.

551. *Sur les espaces à structure uniforme et sur la topologie générale.* By A. Weil. Publications of l'Institut Mathématique de l'Université de Strasbourg, 1938. 40 pages. 15 fr.

Enciclopedia della Matematiche Elementari. Volume II, Parte II. A cura di L. Berzolari, G. Vivanti e D. Gigli. (Biblioteca Matematica.) Milano, Ulrico Hoepli, Editore Librairo della Real Casa, 1938. 11+573 pages. 75 lire.

Introduction to College Mathematics. By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1938. 12+373+93 pages. $2.40.

Leitfäden der Algebra. By H. Stohler. Dritter Teil. (Mathematisches Unterrichtswerk für höhere Mittelschulen. Leitfäden und Aufgabensammlungen. Herausgegeben vom Verein Schweizerischer Mathematiklehrer.) Tabellen zu Stohler, Algebra Leitfäden, Dritter Teil, und Stähli und Meyer, Algebra Aufgabensammlung, Dritter Teil. (Unterrichtswerk des Vereins Schweizerischer Mathematiklehrer.) Zurich and Leipzig, Orell Füssli, 1938. 158 pages. Halbleinen Fr. 3.60. RM 2.20.

The Kelley Statistical Tables. By T. L. Kelley. New York, The Macmillan Company, 1938. 136 pages. $4.50.

The Reverse Notation. By J. H. Johnston. Introducing negative digits with twelve as base. London and Glasgow, Blackie and Son Limited, 1937. 10+74 pages. 3s. 6d.

Statistical Methods Applied to Economics and Business. By F. C. Mills. Revised Edition. New York, Henry Holt and Company, 1938. 19+746 pages. $3.75.

Corporations and their Financing. By Hastings Lyon. New York, D. C. Heath and Company, 1938. 7+946 pages. $4.25.

College Algebra. By W. L. Hart. Revised Edition. New York, D. C. Heath and Company, 1938. 8+408+30 pages. $2.24.

REVIEWS

Les Géométries. By Lucien Godeaux, Paris. (Collection Armand Colin No. 206) 1937. 216 pages. Bound 17.50 fr. Unbound 15 fr.

The little volume by an author well known for his studies in algebraic geometry, admirably achieves its modest aim of giving a brief general view of the history and nature of elementary geometry in language that may be significant to the non-professional reader. The trained mathematician need not expect to find here new theorems or unfamiliar philosophy. The work falls into six chapters: I. Elementary geometry, II. Analytic geometry, III. Projective geometry, IV. Principles of geometry, V. Geometry and the theory of groups, VI. Topology. Chapter IV discusses postulates, starting with the parallel postulate, and then taking up the postulate of Archimedes, the "fundamental theorem" of projective geometry, and duality. In Chapter V euclidean affine and projective transformations are considered in the light of group theory. An Arguesian space is defined as obtained from cartesian space by adjoining points at infinity. This chapter gives the theorem: The ruled projective geometry of ordinary space is equivalent to the geometry of a hyperquadric of five-dimensional projective space having as principal group the group of homographies of this space transforming this hyperquadric into itself. In the chapter on topology such topics as knots, the Peano curve, infinitesimal geometry, and the polyhedral formula are discussed. This book is not a textbook. It offers much not usually given in courses. It reads fluently, and avoids vague allusions to more difficult fields. This work should prove of service and inspiration to the inquiring student, particularly if he is seeking material for reporting in a mathematics club meeting and in any case if he is planning to teach in the elementary schools.

A. A. Bennett

Aspects of Science. By Tobias Dantzig, New York, The Macmillan Company, 1937. 280 pages. $3.00.

The recent appearance of *Aspects of Science* was no doubt greeted with gladness by all those who had had the good fortune to read Dantzig's earlier volume, *Number, the Language of Science.* To say that the new book is on the same high level as the older one is high praise but well deserved. Both of the books are enlightened mathematically, scientifically, and philosophically. Both are wide in their appeal, being charmingly written in a style distinguished for its clarity, force, and magnanimity. Technical symbols are not present in sufficient force to frighten or repel the general reader. Neither can the narrowest specialist reasonably complain. For, though Dantzig neither eschews generalities nor disdains the graces of literary expression, he yet attains so high a degree of precision that not even the illiberal taste of a sheer technician is likely to be offended. I know of no teacher or mathematician or scientist or philosopher or student of letters who could not improve his attainments and outlook by an attentive reading of these books.

The point of view maintained throughout *Aspects of Science* is large, revealing, and thoroughly humane. It "views the evolution of scientific thought" as the outcome "of man's efforts to resolve the perplexities of human existence," to find himself and his place in the boundless complex of Nature. While the work is, therefore, not primarily devoted to the history of science and is not offered as a contribution to that field, yet a good deal of science history has been necessarily included. And if the book contained nothing but Dantzig's superb account of Galileo, it would be worth far more than the selling price. Not elsewhere, I believe, can one find a better portrait of Galileo the man, intense lover of life; or a better portrait of Galileo the genius, chief among the founders of experimental science; or a better portrait of Galileo the victim of the Inquisition. Many a reader will wish to thank the author for reminding him so vividly of what no devotee of science should ever forget—the Dread Antinomy between the authority of Reason and that of organized official Superstition.

But Galileo is only one among a host of creative pioneers, ancient, medieval, and modern, whose contributions are happily signalized in telling the fascinating tales of the growth and developments of geometry, of the number concept and analysis, of experimental science, and of the progressive mathematicization of experience. In this connection it is a pleasure to note that the roll of Dantzig's heroes is far from being a mere list of great scientists and mathematicians. Philosophers, too, are represented, and poets with other men of letters, and the cultural significance of the book is due in no small measure to the wisdom which these have contributed to its pages.

In a few lines it is, of course, quite impossible to analyze the work or to examine its abstract discussions with respect to questions of thoroughness, validity, and conclusiveness or to evaluate the wealth of its meditations. Even if there were room for it, the trial here would only be preliminary, for judgment in such cases belongs ultimately to the jurisdiction of the reader.

I cannot refrain, however, from pointing out that among the book's dozen chapters there are three which have impressed me as especially noteworthy for their lucidity, their solidity, and their penetration. They are respectively entitled "The Infinite," "The Crisis," and "In Quest of the Absolute."

Dantzig's discussion of the vexed question of mathematical infinity is acute and, in my opinion, sound. Its conclusion may be virtually summarized by the dictum: No infinity, no mathematics.

When he says that "the future historian may call our period *the great crisis*," he is not thinking exclusively or even mainly of the much talked-of revolution in physics. He is thinking of something much vaster, something of which that revolution is, properly speaking, not the cause but only one among many symptoms. For what he is thinking of is the deep, swift, multitudinous, uncontrollable changes occurring everywhere in the realm of human life—in the whole world of human thought, human emotion, human sentiment, and human will. Of *that* crisis what are the causes and what the implications? Such are the questions daringly grappled with in chapter six.

The cardinal enterprises of man—art, religion, philosophy, science—are but differing forms or aspects of the human spirit's age-old quest for the eternal, for what abides unchanged. The answers have been innumerable, and all of them false. From time immemorial the wake of the great ship of Thought has been strewn with abandoned absolutes, and still the ship sails on.

Why not? Koheleth may declare that "all is vanity and chasing after wind" and yet admonish: "Fear God and keep his commandments, for this is the whole duty of man."

In like spirit concludes Dantzig: "Read your instruments, and obey mathematics, for this is the whole duty of the scientist."

C. J. KEYSER

Portraits of Eminent Mathematicians, with Brief Biographical Sketches. By David Eugene Smith. New York, Scripta Mathematica, Portfolio Number Two, 1938. 13 portraits. $3.00.

Persons who have seen the first portfolio of the present series published by Scripta Mathematica need no recommendation of these portraits. Those who have not seen it can find a description on page 39 of the present volume of this MONTHLY. In both portfolios portraits are reproductions of pictures in the David Eugene Smith collections and the biographies are by him. Portraits of the following men make up Portfolio Number Two: Euclid, Cardan, Kepler, Fermat, Pascal, Euler, Laplace, Cauchy, Jacobi, Hamilton, Cayley, Chebishef, Poincaré.

I have only praise for the present work, as for its predecessor published in 1936. Portraits published by Professor Smith some years ago are now a classic in American mathematics. There is some duplication with the present series. All the pictures are suitable for framing and will make a handsome addition to any library or classroom.

TOMLINSON FORT

Magic Squares of $(2n+1)^2$ *Cells.* By M. J. van Driel. London, Rider and Company, 1936. 90 pages. $2.75.

As the title indicates, the book is concerned with magic squares of odd order only. The principal method used for constructing magic squares is essentially that of combining two latin squares, a method adopted by Bachet as long ago as 1624. However, by a skillful arrangement of the work the author is able to obtain some very interesting results. One of these, although not new, is the enumeration of all uniform step squares of order five by very elementary means. It should be pointed out in this connection that the meaning of the term "uniform step square" as used in the book is more general than the sense in which it is commonly used. The author includes under that term not only the usual uniform step squares but also those derivable from them by the method of substitutions.

The construction of pandiagonal squares of order 15 in Chapter 2 is note-

worthy since the uniform step process breaks down for pandiagonal squares whose order is divisible by 3. Chapter 3 contains a brief discussion of symmetric magic squares and Chapter 4 is devoted to the so-called rectangular, and square ply magic squares. A magic square is said to be $a\times b$ ply, where a and b are factors of the order of the square, if the sum of the numbers in every rectangular compartment of dimensions a cells by b cells is the same no matter from what part of the original square the compartment be chosen. If $a\neq b$ the long axis must be taken either horizontally or vertically throughout; it cannot in general be taken both ways in the same square. The author points out the interesting fact that uniform step squares of order 9 which are 3×3 ply are necessarily pandiagonal. In Chapter 6 it is shown that non-uniform step squares of order 9 and 3×3 ply may or may not be pandiagonal.

The fifth chapter contains a further discussion of symmetric squares and treats bordered magic squares in some detail. It is unfortunate that in the references to the literature on the latter type of squares the fundamental work by B. Violle published in 1838 is not mentioned.

In the sixth and last chapter some interesting squares of orders 9 and 15 are constructed by combining certain auxiliary squares having magic properties. The method may be used to construct simple squares or squares having the additional properties of being pandiagonal, symmetric, or $a\times b$ ply. The analysis is presented in such a way as to make clear how several of those properties may be obtained simultaneously in the same square.

The book concludes with a summary of four pages, written in French.

The author's limited command of English idiom occasionally leads to some confusion. In the paragraph just below the magic squares on page 10 the phrase "rows of numbers" occurs and is frequently used throughout the remainder of the book. Since in this connection the word "rows" does not refer to the rows of a magic square the reviewer suggests that this phrase be replaced by the expression, "sets of numbers." Also the meaning of the term "headlines" of sections 50 and 51 is perhaps better expressed by the phrase "principal lines."

The casual reader, looking merely for some diversion, is apt to find the book hard going. However, a reader willing to dig will turn up a lot of interesting material; sufficient to keep him occupied for some time.

G. E. Raynor

The Elements of Analytic Geometry. By H. E. Buchanan and G. E. Wahlin. New York, Farrar and Rinehart, 1937. 9+256 pages. $2.25.

This new textbook is a thorough course in the topics ordinarily considered essential in a study of plane analytic geometry (166 pages) and solid analytic geometry (64 pages). The type is fairly small so that it seems probable that the text could be used for as much as a five- or six-hour course if desired. Although this is the first edition of the text, it is singularly free from typographical errors.

In common with many analytic geometry texts, the conic sections (they are proved to be such) are treated in great detail. The general conic is first defined,

in terms of eccentricity, and particular properties of the specific types of conics are developed later. The authors give an excellent discussion of the general equation of the second degree. The transformed equations, following a translation or rotation, are calculated from the necessary invariants; and the usual method of determining the equation by direct substitution is also given.

Various topics, including the determination of the type of conic from the discriminant of its equation, are given as starred exercises rather than as part of the main body of the text. There is an abundance of well graded exercises, with the starred ones to provide additional stimulus for the better students. Included in these latter are suggestions for class reports on historical topics. Answers are given in the back of the book for all problems.

Some of the following topics which appear in this text are not always included in others:

(1) Emphasis upon directed line segments.

(2) A brief discussion of limits and the definition of a derivative.

(3) A chapter on empirical equations.

(4) A discussion of cylindrical and spherical coördinates.

(5) A mention of curves used to trisect angles and to duplicate the cube.

Some instructors will regret that:

(6) There is almost no use made of determinants.

(7) There is no attempt made to make the presentation inductive.

(8) Emphasis appears (in the reviewer's opinion) to be upon the results and derived formulas rather than upon the methods of procedure. For example, on page 118, after obtaining six forms of equations of tangents to conics by using derivatives, the authors state: "The above formulas should be memorized."

(9) Some definite statements are made which are matters of opinion. On page 90, we read, "If the real intercept of the hyperbola is on the y-axis, it is better to take the equation in the form

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1, \cdots ."$$

(10) In discussing geometry of three dimensions, there is almost no attempt at correlation with the analogous procedures and concepts of two dimensions.

In the preface the authors state; "Every mathematics instructor is aware of the urgent need . . . to impart to the average student some feeling for mathematics as a living subject, one that has not only had a tremendous influence on the development of our civilization, but is vitally important in present-day affairs." Whether or not *any* text in analytic geometry, including this one, has satisfactorily filled this need is still questionable. Professors Buchanan and Wahlin have, however, written a text including fairly standard material but with enough variations in method to stimulate, possibly, new viewpoints for the instructor who adopts it.

E. R. Ott

Advanced Calculus. By W. B. Fite. New York, The Macmillan Company, 1938. 12+399 pages. \$5.00.

As the author states in his preface, "this book has been written to supply an introductory course in mathematical analysis for those who are looking forward to specializing in mathematics." To the reviewer it seems that the stated purpose has been attained in a satisfactory manner. Assuming familiarity with the working rules and simpler applications of the calculus, the first part of the book discusses the meanings of the fundamental concepts of the derivative and integral together with proofs of the fundamental theorems. The selection of subject matter in the book conforms with that which has now become somewhat standardized for a course in advanced calculus.

The book opens with a chapter on "the system of real numbers," giving a brief treatment of real numbers on the Dedekind basis; in the last four articles of the chapter the complex numbers are introduced as number-pairs. The second and third chapters treat functions of one and of several variables respectively, and ordinary and partial derivatives, including the directional derivative. The following chapter is on Taylor's expansion with a remainder, for functions of one and of several variables, with applications to indeterminate forms and maxima and minima; it includes articles on Euler's constant and Lagrange's multipliers.

The next four chapters are devoted to integration. The first of these treats the definite integral of a function of one variable, starting with a geometric approach and then giving an existence proof, covers the usual topics here, including the mean value theorems, differentiation under the integral sign, and approximate integration by Simpson's rule with the determination of the error by this rule. Then follows a chapter on indefinite integrals, dealing with integration of rational fractions in general terms and a brief introduction to elliptic integrals. The chapter on improper and infinite integrals includes tests of convergence, the Gamma and Beta functions and Stirling's formula and series. While Chapter VIII is entitled "Double and Triple Integrals," it also contains a discussion of line and surface integrals, with Green's and Stokes's theorems.

The following group of three chapters is devoted to series. The first of these, on "Infinite Series," gives a fairly extensive treatment for such a book as this. It covers the usual topics on tests of convergence, series of variable terms and uniform convergence, and double series, and includes an article on Weierstrass's example of a continuous function without a derivative. The reviewer was pleased to see here an adequate definition of the term "infinite series." Then follows a chapter on power series, covering the usual topics. Chapter XI is on trigonometric series and series of orthogonal functions; the treatment here is more extensive than in other advanced calculus texts.

The next two chapters are concerned with implicit functions and functional determinants, and applications to geometry, including applications to plane curves, skew curves, and surfaces.

In the last two chapters a brief introduction is given to the calculus of varia-

tions and to the theory of functions of a complex variable. In the former there is an interesting application of the calculus of variations to a problem in economics in addition to the usual applications to geometry and physics.

There is sufficient material in the book for a year's work. The style is clear, and the treatment is perhaps as rigorous as is possible for the average student at this stage of his development.

L. L. SMAIL

Introduction to Mathematical Probability. By J. V. Uspensky. New York and London, McGraw-Hill Book Company, 1937. 9+411 pages. $5.00.

As stated by the author at the outset, the book begins in a truly elementary fashion, and expands in content to include much of recent work by such writers as Liapounoff, Markoff, Tshebysheff, Khintchine, Kolmogoroff, and "Student." By thus making available such a collection in English for the first time, Uspensky's book furnishes a fitting sequel to the final remark of Struik in his review of Fréchet's, *Généralités sur les Probabilités* (*Bulletin of The American Mathematical Society*, vol. 43, p. 603).

A partial list of topics treated will be of assistance in indicating the large scope of the text. They are: definitions, repeated trials, probabilities of hypotheses, difference equations and simple chains, Bernoulli's theorem, approximate evaluation of probabilities, mathematical expectation, law of large numbers, probabilities in continuum, general concept of distribution, fundamental limit theorems, normal distribution in two dimensions, method of moments. All through the text and particularly at the end of each chapter the theory is illustrated and many times introduced by means of an abundance of examples. From a teacher's standpoint the reviewer feels that this book is an exceptionally good one for giving self-reliance to students, through the interesting theory problems which have just enough hints to encourage them to actually attempt solution. The serious student of probability will find this a fine coördinating study for numerous topics of algebra, real and complex variable, *e.g.*, Lagrange's solution of finding the probability of exactly x successes in t independent trials with constant probability p, page 87.

Through the study of the composition formula for two distribution functions there is given in Chapter XIII the proof that the sum of normally distributed independent variables is again normally distributed with variance equal to sum of variances. In the following chapter one finds the rather long proof of Laplace's general theorem that the sum of n independent variables (with mild restrictions) divided by standard deviation of the sum is nearly normal if n is large. Of course one cannot include a proper statement of all the comparably relevant facts treated in this volume.

Only a few typographical errors were observed such as on pages 42, 94, and they would occasion no difficulty, being obvious. The book is evidently the work of an inspiring teacher and will undoubtedly take an important place among reference books and textbooks of probability.

J. A. GREENWOOD

Actualités Scientifiques et Industrielles. Paris, Hermann et Cie, 1936.

325. *Cinématique du solide et théorie des vecteurs*. By Ch. Platrier. 54 pages. 12 fr.

326. *La masse en cinématique et théorie des tenseurs du second ordre*. By Ch. Platrier. 81 pages. 18 fr.

327. *Cinématique des milieux continus*. By Ch. Platrier. 34 pages. 8 fr.

427. *Les axiomes de la mécanique newtonienne*. By Ch. Platrier. 59 pages. 14 fr.

The total of 169 pages in the first three of these little books constitute a good account of kinematics, suitable for an advanced course in mechanics. For the most part the material treated is of a classic nature and the exposition resembles what is found in the larger treatises. Nevertheless, the author has given a freshness and a liveliness to his account which make his work readable and interesting. He has also organized the material in such a way that theorems and propositions stand out with proper emphasis. The attentive reader should bring from the work clearly defined and usable ideas, though the absence of exercises prevents the attainment of the mastery which comes through intimate practice. The discussion of vectors and tensors which is contained in the work is quite limited, being restricted to the actual needs of the applications made.

The printing is of the high order that characterizes the useful series of monographs of which these are members.

In the fourth pamphlet one finds a discriminating and clear discussion of the axioms of classical mechanics. Every teacher of mechanics is aware of the perplexities and difficulties that arise when one seeks to present a foundation for the subject. Many of the terms used constantly throughout a course are inadequately understood by students. There has been for some time an extensive and critical literature dealing with the foundations of mechanics, but much of it requires considerable maturity and extensive study for its appreciation. The work under review meets a real need and instructors should be glad to have it available.

This monograph begins with a consideration of the ideas of solids, of periodic phenomena, and of quantity of matter; gives a discussion of the common basic units and of derived units; discusses homogeneity and similitude; sets forth fundamental axioms (giving with proper dates 28 names of men associated with the fundamental principles of mechanics, from Aristotle and Archimedes to Hertz and Painlevé); discusses such questions as the identity of gravitational mass and inertial mass; and ends with a consideration and classification of forces.

Although the work deals entirely with classical mechanics, references are made to relativity theory, and it is pointed out that relativity rejects certain postulates of classical mechanics when velocities comparable to those of light are being considered.

K. P. Williams

Bibliographia Kepleriana. A guide through the printed writings of John Kepler, with 80 facsimiles. Published by Max Caspar in collaboration with Ludwig Rothenfelder for the Bavarian Academy of Sciences. Munich, 1936, 158 pages, 79 plates.

This scholarly bibliography of the works of Kepler is based in the first instance upon an examination of the publications themselves in central European libraries. Kepler himself prepared in 1622 a list of his publications which is reproduced in facsimile. Fourteen other bibliographical lists published between 1711 and 1931 have been utilized by the author, and doubtless also that given by Poggendorff which is not cited despite its importance. To these bibliographies must be added as a source of information the *Joannis Kepleri Astronomi Opera Omnia*, Frankfort and Erlangen, 1858–1871, edited by Christian Frisch.

The importance of the present publication rests upon the complete bibliographical apparatus, including location of copies and reproduction of the title pages of all the known original editions and even the editions published during the author's lifetime. The bibliography prepared by Mr. Frederick Brasch for the History of Science Society Publication, *Johann Kepler, 1571–1630, A Tercentenary Commemoration of His Life and Work* (Baltimore, 1931), is properly praised as giving a much more complete list than the earlier ones. However, this list does not give collation or location of copies in America.

The usefulness of the bibliography would have been increased by the inclusion of the holdings of the Bibliothéque Nationale in Paris and the British Museum, both of which have notable collections of Kepleriana. However, with the bibliography it is comparatively easy to place the copies, given in the respective catalogues of the two libraries.

America has, so far as I know, no outstanding collection of Kepler's early works. Probably the John Crerar Library of Chicago has more of the original editions than any other American library. An edition there of the *Gründtlicher Bericht von einem ungewönlichen newen Stern Welcher im Oktober disz 1604 Jahrs erstmahlen erschienen.* (Prag. . . . Anno M.DC.V.) is a variant of 1605, not given by Caspar among the six variants of this title issued in the period 1604 to 1605.*

For the history of mathematics as well as astronomy and even physics, Kepler has superlative importance. This has been well indicated by D. J. Struik in his article, *Kepler as a Mathematician*, in the above-mentioned American Tercentenary publication on Kepler. To the evolution of the differential and integral calculus Kepler contributed fundamental material touching the most diverse phases of applications of the infinitesimal reckoning which connect closely with the most outstanding developments of the new universe of mathematics and of astronomy.

L. C. KARPINSKI

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* I am indebted to J. C. Bay, Librarian of the John Crerar Library, for confirmation of this title as given in the Union Catalogue.

Miscelanea Matematica. By J. Barinaga. Madrid, Nuevas Graficas, 1937. 4+127 pages.

The remarkable vitality of the Spanish people is indicated by a publication of this stimulating type created during such tragic times. The author is Professor of Mathematics in the University of Madrid and Director of the Mathematical Laboratory.

To many readers the biographical notices and notes which form a large part of the work will prove of particular interest. The dozen portraits—Ernst Steinitz, Otto Schreier, Jacques Herbrand, Cesáro, Riemann, Georg Cantor, Jacobi, Kelvin, Kronecker, Peano, Hilbert, and Abel—indicate the variety of fields touched and often illuminated in this volume. Largely the theoretical notes concern algebraic theory.

The Spanish students of mathematics today are fortunately not deprived of current material presented in an interesting way by scholars with broad, mathematical training and wide historical interests. The international character of the publication is worthy of comment; the portraits and biographical notes include Italian and German but no Spanish mathematicians; the author does not find it necessary to take a narrow, nationalistic attitude on scholarly matters.

More than 400 names appear in the Index, giving of living authors the date of birth, and of others the date of birth and of death.

Teachers will find in this work many practical suggestions for inspiring students to examine and to read the great classics of mathematical literature.

L. C. KARPINSKI

A CORRECTION

In the review of *Introduction to mathematics* by Cooley (*et al*) in the March (1938) issue of this MONTHLY the price of the book was incorrectly given. The price is $3.25 and not $3.75 as stated. E.J.M.

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics, with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

The new editor wishes to take this opportunity to thank Professor and Mrs. Owens in behalf of all of our Mathematics Clubs for the real help and inspiration they have been to us in conducting our work.

CLUB TOPICS

In January 1918, Professor R. C. Archibald of Brown University inaugurated this department of Mathematics Clubs. To him we are indebted for the beginning of suggested topics for club programs and accompanying bibliographies. These should be as helpful to our friends who prepare papers for mathematics clubs today as they were at that time. And so, for your convenience, we are making a reference list of many of these topics to which we hope, in the course of time, to make additions. Surely there are Monthly readers who have in their files fairly complete bibliographies on topics suitable for mathematics clubs which they would be willing to contribute to this department.

The following is a list of papers which have appeared in this Monthly.

The oldest mathematical work extant, vol. 25, p. 36.

Geometrography and other methods of measurement of geometrical constructions, vol. 25, p. 37.

Arithmetical prodigies, vol. 25, p. 91.

Ptolemy's theorem and formulae of trigonometry, vol. 25, p. 94.

Paper folding, vol. 25, p. 95.

Women as mathematicians and astronomers, vol. 25, p. 136.

The binary scale of notation, a Russian peasant method of multiplication, the Game of Nim, and Cardan's Rings, vol. 25, p. 139.

The logarithmic spiral, vol. 25, p. 189.

Golden section, vol. 25, p. 232.

A Fibonacci series, vol. 25, p. 235.

Euler Integrals and Euler's Spirals, sometimes called Fresnel Integrals and the Clotholde or Cornu's Spiral, vol. 25, p. 276.

Geometry of four dimensions, vol. 25, p. 316.

Constructions with a double edged ruler, vol. 25, p. 357.

The cattle problem of Archimedes, vol. 25, p. 411.

The number π, vol. 26, p. 209.

Codes and ciphers, vol. 26, p. 409.

The Chinese suan pan, vol. 27, p. 180.

Finite geometries, vol. 28, p. 85.

Functional equations, vol. 32, p. 428.

Fiedler's cyclography, vol. 32, p. 517.

The pasturage problem of Sir Isaac Newton, vol. 33, p. 155.

Theorem of Bang, isosceles tetrahedra, vol. 33, p. 224.

La courbe du diable, vol. 33, p. 273.
Additional references to club topics, vol. 43, p. 40; vol. 44, p. 537.
Calculating machines, vol. 43, p. 99.
Development of present day numerals, vol. 43, p. 99.
Four color problem, vol. 43, p. 181.

INTERCOLLEGIATE MATHEMATICS ASSOCIATIONS

The reports of club activities which have come to this department show an increasing interest in the formation of intercollegiate mathematics clubs in metropolitan areas. It would seem that such organizations might well justify their existence in promoting a friendly feeling among college students with the same major interests, encouraging a more ambitious mathematical program, stimulating research and developing forums which make possible the discussion of problems arising in separate clubs.

Perhaps, as time goes on, some of these organizations will find it possible to extend their contacts to include all of the mathematics clubs of their particular state or section of the country for at least one meeting of the academic year. It might even be possible to schedule this meeting at the same time as the state meeting of the Mathematical Association, so that professors and students might well be in conference at the same time.

This department would be interested in hearing of other organizations of this type. If you are promoting a particularly ambitious program, which you think would be suggestive to other clubs, we should very much appreciate a full account.

CLUB REPORTS
1937–38

Intercollegiate Mathematics Association, Milwaukee

This association held four meetings in the course of the academic year. The program included: at Milwaukee State Teachers College, "Life probabilities" by Mr. Fassel of the Northwestern Mutual Life Insurance Company; at Marquette University, an informal social meeting featuring a play entitled "Chums"; at Mount Mary College, five talks, one from a representative of each of the participating clubs, on the topics: Ideas of infinity and Lucretius; Practical mathematics in Roman Times; Mathematical fallacies; The geometric proof and Work of the C.C.C.; at University of Wisconsin Extension Division, the annual banquet with an address by Mrs. William Beckwith of Milwaukee Downer College.

President, Frederick Adler, M. U.; Vice-President, Margaret Kreiziger, M. M. C.; Secretary, Gordon Voigt, U. W. E. D.; Treasurer, Elmer Poppendieck, M. S. T. C.; Corresponding Secretary, Norma Fedders, M. D. C.

Pythagorean Club, Milwaukee State Teachers College

Topics discussed at the bi-monthly meetings were: Fundamental concepts of mathematics; Relation of mathematics to art; Place of mathematics in the curriculum; Geometric fallacies; How to teach the addition and subtraction of negative numbers; Origin of numerical forms. Social meetings of the year featured a play "A Mathematical Nightmare," spelldown using mathematical terms, a chili supper and a picnic.

President, Laura Gilbert; Vice-President, Elmer Poppendieck; Secretary, Jennie Brodi; Treasurer, Anna Goepfert; Faculty Advisor, Elizabeth Knight.

Junior Mathematics Club, University of Wisconsin, Extension Division

Each year this club entertains junior and senior high school students from Milwaukee and suburbs. At this year's meeting, motion pictures on geometry and astronomy were shown; two students spoke on "Mayan Number System" and "The Moon" respectively. An exhibit included calculating machines, astronomical models and instruments, slide rules, string and plaster and wooden models, charts, drawings and problems.

The Euler Prize awarded annually for the best paper presented on a mathematical subject was won by Yvonne Town with a paper entitled "Arithmetic based on Number Systems Other Than Ten."

President, Gordon Voigt; Secretary-Treasurer, Aimee Schultz; Faculty Advisor, Louise A. Wolf.

Mathematics Club, Milwaukee Downer College

Subjects for programs: Theory of relativity, Linkages and string figures, Mazes and labyrinths, Duo-decimal number system, The solar system and Meteorology. Social meetings: introductory meeting, a mathematical treasure hunt; Christmas meeting, mathematical bibliography; close of the year, a banquet.

President, Elizabeth Little; Secretary-Treasurer, Mary Frances McKee.

Mathematics Club, Mount Mary College, Milwaukee

This club sometimes met jointly with the Science Club of the same school. Talks at meetings included one by Dr. J. D. Ball on Dr. Steinmetz, another on "The mathematical education of women during the middle ages." At the annual party former members who had graduated were guests.

President, Margaret Kreiziger; Vice-President, Marion Clark; Secretary-Treasurer, Mary C. Neugent.

Joint Meetings in the Greater Boston Area

Three joint meetings have been held in Boston. The first was attended by members of the Boston University and Wellesley College Clubs. Tufts College club members met with the group at their second and third meeting. Each college group was the host for one meeting. Most of the talks at these meetings were made by students. At the last meeting, Professor Philip Franklin of Massachusetts Institute of Technology spoke on "Four color maps." Matching games were used to help members to get acquainted.

Plans are now being made to include the clubs at Northeastern College, Boston College and Massachusetts Institute of Technology as well as the above mentioned clubs. Two meetings are scheduled for next year, the first to be held in October at Boston University. The presidents of the clubs form the executive committee of the new society.

The Mathematics Club, Wellesley College

Subjects discussed at formal meetings were: Workers in the theory of numbers; Number systems; Probability and calculus.

President, Grace Mandeville; Vice-President, Evelyn Wicoff; Senior Executive, Doris Gasteiger; Junior Executive, Ruth Hawkes; Secretary, Gloria H. Sharp.

Mathematics Society, Northeastern University, Boston

Guest speakers at the monthly meetings of this organization were: Professor Elmer E. Haskins on "Use of series" and Professor Raymond K. Morley of Worcester Polytechnic Institute on "Polar coördinates," (using gear machines to demonstrate certain curves). The other meetings were devoted to the three famous topics of antiquity; Dimensional analysis; Mathematical games and puzzles; and the synthetic and analytic methods of solving the problem:—To find the point in the plane of any given triangle such that the sum of the distances from the point to the vertices of the triangle shall be a minimum.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 339. *Proposed by V. Thébault, Le Mans, France.*

Consider all the triangles inscribed in a given circle on a fixed chord as base, and determine the locus of the feet of the interior and exterior base-angle bisectors. Thus show that if two interior angle bisectors of a triangle are equal, the triangle is isosceles.

E 340. *Proposed by E. C. Kennedy, Texas College of Arts and Industries.*

If B is a positive integer, what rational values may be assumed by

$$R = \sqrt{\left(\frac{B}{7}\right)^2 + 5038}$$

E 341. *Proposed by J. H. Edmonston, Washington, D. C.*

A triangular octahedron (one with all faces triangular) may be regarded as a space analogue of a plane quadrilateral. On this basis, state and prove a space-analogue of the theorem that the midpoints of the sides of any plane quadrilateral are the vertices of a parallelogram.

E 342. *Proposed by W. E. Buker, Pittsburgh, Pennsylvania.*

Show how to draw three circles with radii a, b, and c, and common tangents d, e, and f.

E 343. *Proposed by H. E. Stelson, Kent State University.*

Prove that $ax^2+2hxy+by^2+2gx+2fy+c$ is the product of two factors, linear and rational in x and y, provided $[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}]-b[g+(g^2-ac)^{1/2}]=0$, and that in this case, $abc+2fgh-af^2-bg^2-ch^2=0$.

E 344. *Proposed by V. W. Graham, High School, Dublin, Ireland.*

$ABCD$ is a parallelogram with vertices named in order around the perimeter. DA and CB are produced to P and Q respectively, so that $AP=BQ$. Any point X is taken on AB. PX meets the diagonal BD at Z. QZ meets DC at Y. Prove that $AX=DY$. Conversely, if $AX=DY$, prove that Q, Y, and Z are collinear.

CORRECTIONS. There was an error made in the statement of E 310 [1938, 47], which should read:

E 310. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $aabb$ which is the square of bb. Show that the numbers, b and $a^2+4(a-1)^2$ are perfect squares. Determine the base of such a system and the values of a and b, knowing that a is also a perfect square.

In line six on page fifty-two of the January issue (1938) the word "perimeter" should read "semi-perimeter."

SOLUTIONS

E 208. *Proposed by G. A. Whittemore, New York City.*

How may eight married couples play a seven-round bridge tournament, if each man plays one round with each lady except his wife, and everybody plays against everybody else except his or her spouse? How many solutions exist? Can this be generalized for $4n$ couples playing $4n-1$ rounds? Does a solution exist for $4n+2$ couples playing $4n+1$ rounds?

Partial Solution by W. E. Buker, Pittsburgh, Pennsylvania.

H. E. Dudeney, on page 203 of his *Amusements in Mathematics*, gives a solution of this problem for four couples. He notes that a solution for eight couples exist, and calls it a hard puzzle, but gives no solution.

Here is a solution:

Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
$Ab-De$	$Ac-Ef$	$Ad-Fg$	$Ae-Gh$	$Af-Hb$	$Ag-Bc$	$Ah-Cd$
$Cg-Hf$	$Dh-Bg$	$Eb-Ch$	$Fc-Db$	$Gd-Ec$	$He-Fd$	$Bf-Ge$
$Fh-Gc$	$Gb-Hd$	$Hc-Be$	$Bd-Cf$	$Ce-Dg$	$Df-Eh$	$Eg-Fb$
$Ed-Ba$	$Fe-Ca$	$Gf-Da$	$Hg-Ea$	$Bh-Fa$	$Cb-Ga$	$Dc-Ha$

Cyclic permutation of each of the letters yields seven new solutions.

E 304 [1937, 659]. *Proposed by Michael Goldberg, Washington, D. C.*

Superimpose a given circle upon a given polygon so that their common area is a maximum. Show that the segments which the circle intercepts on the sides of the polygon can form a closed polygon in which the directions of the segments are also preserved. Locate the center of the circle. When the polygon is a triangle, determine the locus of the center of the circle as its size is increased from that of the inscribed to that of the circumscribed circle.

Solution by the proposer.

Let the lengths of the sides of the polygon be a_i and the intercepts on them be q_i. If the circle is moved a distance dx in any direction, then the change in the area common to circle and polygon is $\sum q_i \sin A_i dx$ (except for infinitesimals of higher order) where the A_i are the angles which the sides of the polygon make with the direction of dx. For maximum area, $\sum q_i \sin A_i dx=0$, or $\sum q_i \sin A_i=0$.

But since this equation must hold for every direction of dx, the q_i must be equivalent to a set of balanced vectors. That is, they can form a closed polygon whose sides are parallel to the sides of the given polygon.

In particular, if the given polygon is a triangle, the polygon of the q_i is a similar triangle, and the q_i are proportional to the sides of the original triangle, a_i.

E 305 [1937, 659]. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

If the external angle bisectors at A and B are equal, must the triangle ABC be isosceles?

Solution by W. E. Buker, Pittsburgh, Pennsylvania.

Let angle A be a right angle, let the bisector of external angle B meet CA produced at P, and the bisector of external angle A meet BC produced at Q.

In triangle $BAP, BP=c/(\sin B/2)$. In triangle $BAQ, AQ=c\sin B/\sin(45°-B)$. Now if $BP=AQ$, then $c/(\sin B/2)=c\sin B/\sin(45°-B)$, or $\sin^2 B+\cos B=2\sin A$. This equation is satisfied by a value of B slightly greater than 35°, so that the external angle bisectors of a triangle may be equal without that triangle being isosceles.

Also solved by E. P. Starke and the proposer.

E 306 [1937, 659]. *Proposed by W. B. Campbell, Ithaca, New York.*

Each license plate in a certain state bears from one to five characters, of which not more than two are chosen from the 26 letters of the English alphabet, the remainder being chosen from the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Any letter used can be in any position. Find the number of different plates possible.

Solution by E. P. Starke, Rutgers University.

If the license plate bears n characters, of which i are letters, the places in which the letters are to appear may be selected in ${}_nC_i$ ways, the letters for those places may be selected in 26^i ways, and the digits for the remaining places may be selected in 9^{n-i} ways. For such plates there are then ${}_nC_i 26^i 9^{n-i}$ possible choices. If this formula is summed for $i=0, 1, 2$, and for $n=1, 2, 3, 4, 5$, we shall have the desired result. It is thus found that there are 35 plates of one character, 1225 of two, 25,299 of three, 410,913 of four, and 5,840,019 of five. Hence in all, there are 6,277,491 possible different plates.

Several other answers were received to this problem, no two of which were alike.

E 307 [1937, 659]. *Proposed by V. Thébault, Le Mans, France.*

Locate the point P in the plane of the given triangle ABC such that the triangles PAB, PBC, and PCA may have equal perimeters.

Solution by the proposer.

It is easily shown that the desired point P is the center of a circle which is

externally tangent to three circles centered at the vertices A, B, and C of the original triangle, and with respective radii of $s-a$, $s-b$, and $s-c$, where s is the semi-perimeter. Furthermore, this point P is also the center of a circle tangent to three circles centered at the vertices A, B, and C of the original triangle, with respective radii of a, b, and c.

It may be further observed that there exists a point Q which is the center of the circle externally tangent to the first set of three circles named above, possessing the property that if it is joined to the vertices of the original triangle, the three new triangles thus formed have semi-perimeters s_1, s_2, s_3 such that $s_1-a=s_2-b=s_3-c$.

Also solved by W. B. Clarke and D. L. MacKay.

E 308 [1937, 659]. *Proposed by E. H. Clarke, Hiram College, Ohio.*

Find the triangle which contains an angle most nearly equal to one radian, from among all possible triangles whose sides are integers of one or two digits.

Solution by R. F. Schnepp, St. Mary's University of San Antonio.

Denote the sides by a, b, and c, and let A, opposite a, be the angle to approximate one radian. Now from the cosine law we have

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{1}$$

Hence $\cos A$ is rational. By means of continued fractions, the convergents to the close approximation, $\cos 1=.54030230$ are found to be 1/1, 1/2, 6/11, 7/13, 20/37, 47/87, 67/124, 181/335, 248/459, 429/794, 20411/37777, $\cdots$. To obtain the most suitable approximation, we choose the convergent farthest to the right that satisfies the conditions of the problem. Since $2bc \cos A$ is an integer, we need not examine convergents with denominators greater than 20,000. This rules out 20411/37777 and all subsequent convergents. The three convergents before that are unable to make the right member of (1) a square.

If we set $\cos A=67/124$, we have $a^2=b^2+c^2-67bc/62$, so that bc is a multiple of 62 and either b or c is 31 or a multiple of 31. While 31 itself will not work, if we let $b=62$ we have $a^2=62^2+c^2-67c$, or $(a+62)(a-62)=c(c-67)$. One obvious solution is $a=62$, $c=67$, and an examination of the c-discriminant shows that it is the only one which meets the given conditions. Hence the sides of the desired triangle are 62, 62, and 67. With these values, the formula $\cos A/2=\sqrt{s(s-a)/bc}$, gives $A=57°17'40''$, whereas 1 radian$=57°17'44''.8$.

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3882. *Proposed by J. R. Musselman, Western Reserve University.*

The pedal circle of the centroid G of a triangle $A_1A_2A_3$ passes through the centers of the hyperbolas of Kiepert and Jerabek. These hyperbolas are considered in Casey's *Analytic Geometry of the Point, Line, and Circle*, pp. 442–448.

3883. *Proposed by J. R. Musselman, Western Reserve University.*

The orthopole of the Euler line of a triangle $A_1A_2A_3$ as to the same triangle is the center of the hyperbola of Jerabek; the orthopole of the Brocard line of the triangle as to the same triangle is the center of the hyperbola of Kiepert. Orthopole is defined in Johnson's *Modern Geometry*, p. 247.

3884. *Proposed by H. S. M. Coxeter, University of Toronto.*

Prove that the points of contact of the real bitangents to a plane quartic (of genus 3) are its points of intersection with 1, 2, 4 or 7 conics. See Baker, *Principles of Geometry*, vol. 6, p. 14 for a theorem similar but not so complete.

3885. *Proposed by V. Thébault, La Mans, France.*

The product of n consecutive positive integers, n being odd, is divisible by their sum, except in the case where, n being prime, the arithmetic mean of the n integers is divisible by n. Examine the case where n is even.

3886. *Proposed by V. Thébault, La Mans, France.*

A parallelogram is inscribed in an ellipse and a point P is chosen arbitrarily on the ellipse. Two straight lines are drawn from P parallel to the sides of the parallelogram cutting them in four points. A third straight line is drawn from P parallel to one of the diagonals of the parallelogram cutting the tangents to the ellipse at the ends of this diagonal in two points. Show that the six points thus obtained are the vertices of a hexagon whose consecutive sides are parallel to two conjugate diameters of the ellipse, and that the area of the hexagon is the same as that of the parallelogram. See 3861 [1938, 122].

3887. *Proposed by V. Thébault, La Mans, France.*

Through the vertex A of a triangle ABC a straight line AM is drawn cutting the side BC in M. Let 2θ be the angle AMC; O and I the centers of the

circumscribed circle (O) and the inscribed circle (I) of ABC. The circles (ω_1) and (ω_2) with centers ω_1 and ω_2 and radii ρ_1 and ρ_2 are each tangent to (O) and the first is tangent also to the two sides of angle AMC while the second is tangent to the two sides of angle AMB. Prove that: (1) The straight line joining ω_1 and ω_2 passes through I. (2) The point I divides the segment $\omega_1\omega_2$ in the ratio $\tan^2\theta:1$; and $\rho_1+\rho_2=r^2\sec^2\theta$, where r is the radius of (I).

SOLUTIONS

3796 [1936, 500]. *Proposed by Don Wallace, Charlottesville, Va.*

Solve the following matric-differential equation using only ordinary methods, i.e., without the use of the matric product integral,

$$\frac{dZ}{dt} + ZA + BZ + ZPZ = 0,$$

where A and B are constant matrices and P is a function of t.

Solution by J. H. M. Wedderburn, Princeton University.

Let $Y=Z^{-1}$, then $dY/dt=AY+YB+P$. Put $C=e^{-At}$, $D=e^{-Bt}$, $X=CYD$; a short calculation gives $dX/dt=CPD$ and therefore $X=\int CPD\,dt$.

Solved also in a similar manner by the proposer.

3797 [1936, 500]. *Proposed by Don Wallace, Charlottesville, Va.*

Show that the locus of points collinear with their isogonal and isotomic conjugates is a conic passing through the median and exmedian points and the incenter and excenters. Show that its center is Steiner's point and that it is a rectangular hyperbola. Show also that the tangents at the median and exmedian points pass respectively through the symmedian and exsymmedian points, and that any point, its isogonal and isotomic conjugates are conjugate points with respect to the conic.

See in this MONTHLY, *On isogonal points*, by J. H. Weaver, 1935, pp. 496–499, and *Isogonal and isotomic conjugates and their projective generalization*, by P. H. Daus, 1936, pp. 160–164.

I. *Solution by Otto J. Ramler, Catholic University of America.*

If we take the lengths of the sides of the reference triangle to be a_1, a_2, a_3, and use trilinear normal coördinates, the isogonal conjugate of (x_1, x_2, x_3) is $(x_1^{-1}, x_2^{-1}, x_3^{-1})$ and the isotomic conjugate is $(a_1^{-2}x_1^{-1}, a_2^{-2}x_2^{-1}, a_3^{-2}x_3^{-1})$. Writing the condition that these points shall be collinear, we obtain in determinant form the equation of the locus of points (x_1, x_2, x_3) which together with their isogonal and isotomic conjugates are collinear,

$$(1)\qquad \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \\ a_1^{-2} & a_2^{-2} & a_3^{-2} \end{vmatrix} = \sum x_i^2(a_k^{-2} - a_j^{-2}) = 0.$$

The equation shows that the fundamental triangle is self conjugate with respect to the conic locus. Another reason for this property will appear later.

The coördinates of the median and exmedian points are $(\pm a_1^{-1}, \pm a_2^{-1}, \pm a_3^{-1})$ and are readily seen to satisfy the equation of the conic, showing that these points lie on the conic. The same may be said of the coördinates of the incenter and excenters $(\pm 1, \pm 1, \pm 1)$. Since the incenter and excenters form an orthocentric set of points the conic must be a rectangular hyperbola. The fact that the sum of the coefficients in the equation (1) is identically zero also shows that the conic is a rectangular hyperbola.

The equation of the polar of (y_1, y_2, y_3) with respect to (1) is

$$\sum a_i^2 x_i y_i (a_j^2 - a_k^2) = 0, \qquad i \neq j \neq k \neq 1, 2, 3. \tag{2}$$

If (y_1, y_2, y_3) lies on the conic, (2) is the equation of the tangent at the point. Simple substitutions show that the symmedian point (a_1, a_2, a_3) lies on the tangent at the median point $(a_1^{-1}, a_2^{-1}, a_3^{-1})$, and that the exsymmedian points lie on the tangents at the exmedian points respectively; for instance $(-a_1, a_2, a_3)$ lies on the tangent at $(-a_1^{-1}, a_2^{-1}, a_3^{-1})$. The center may be obtained as the intersection of the polar of two points at infinity. It is readily shown that the polars of $(0, a_2^{-1}, -a_3^{-1})$ and $(a_1^{-1}, 0, -a_3^{-1})$ intersect at $[a_1^{-1}(a_2^2-a_3^2)^{-1}, a_2^{-1}(a_3^2-a_1^2)^{-1}, a_3^{-1}(a_1^2-a_2^2)^{-1}]$ which is Steiner's point.

Equation (2) is satisfied if $x_iy_i=1$ and if $a_i^2x_iy_i=1$, $i=1, 2, 3$. But these conditions identify (x_1, x_2, x_3) and (y_1, y_2, y_3) as isogonal and isotomic conjugates respectively. Hence any point, its isogonal conjugate and isotomic conjugate are conjugate points with respect to the conic. Since in the isogonal and isotomic transformation the reference triangle is exceptional in that any point on a side corresponds to the opposite vertex, the reference triangle is self-conjugate as to the conic (1). Equation (1) is an analytical confirmation of this fact.

II. *Solution by P. H. Daus, University of California at Los Angeles.*

The transformation by isogonal or by isotomic conjugates is a special quadratic Cremona transformation. Either establishes a 1-1 point correspondence in the plane, except that the vertices and sides are exceptional. To any general point on a side corresponds the opposite vertex, while to a vertex corresponds the opposite side. However, if we associate with each vertex the directions through it, then to each direction corresponds a unique point on the opposite side and conversely. We assume that the 1-1 correspondence has been restored in this fashion and we discuss the locus by considering lines through a vertex of the triangle $A_1A_2A_3$. Let P, P', P'' be the point, its isogonal conjugate, and its isotomic conjugate. Suppose that P moves along a line through the vertex A_2, cutting A_1A_3 in Q, while A_2P' and A_2P'' cut A_1A_3 in A_2' and A_2'', the points which correspond to A_2. Then the point row on A_2Q is projective to those on A_2A_2' and A_2A_2'', with Q corresponding to A_2 on each of the others. Hence the point rows $u'(P')$ and $u''(P'')$ are perspective and the lines $p' \equiv P'P''$ pass through a point, say T, on A_1A_3. If we project P to T by p, we have two super-

posed projective pencils which have two self corresponding lines and for the corresponding positions of P, the points P, P', P'' are collinear. Hence the required locus is such that an arbitrary line through A_2 cuts it in two points T_1, T_2, and hence is a conic c. That it passes through the eight points I_i and G_i, $(i=0, 1, 2, 3)$, is evident since I_i coincides with its isogonal conjugate and G_i with its isotomic conjugate. That the conic is an equilateral hyperbola follows since I_i form an orthocentric system. (See Holgate, *Projective Pure Geometry*, p. 207.)

When the point P takes the position Q, p' cuts A_2Q in A_2, and when P takes the position A_2, p' cuts A_2Q in Q. Hence the projectivity at T considered above is an involution, so that A_2P cuts $P'P''$ in the harmonic conjugate of P with respect to T_1, T_2. The points A_2, Q form a special pair of such points, indicating that the triangle $A_1A_2A_3$ is self-polar with respect to c, since we may take any line through A_2, and also consider the other vertices. In a manner analogous to the above we see that A_3P cuts $P'P''$ in a point which is the harmonic conjugate of P with respect to the points where A_3P cuts the conic c. Hence $P'P''$ is the polar of P with respect to the conic c, that is, any point is the polar conjugate of both its isogonal and isotomic conjugates. In particular, the tangent at any point on c contains the isogonal and isotomic conjugates of the point of contact. This shows that the tangents at G_i pass through the corresponding symmedian or exsymmedian points K_i, $(i=0, 1, 2, 3)$, their isogonal conjugates, and the tangents at I_i pass through their isotomic conjugates.

The property of the center will be determined by the definition of Steiner's point S. For our immediate purpose we define it as the fourth intersection of the circumcircle and the so-called Steiner ellipse (see Sommerville, *Analytical Conics*, p. 59), the isogonal conjugate and isotomic conjugate, respectively, of the line at infinity. Since S' and S'' are both at infinity, it follows that S, the pole of $S'S''$, is the center of the conic c.

Solved also by J. A. Clawson, S. Kaplan, and the proposer.

Editorial Note. The remaining solutions were analytic with the given triangle as the basis of homogeneous coördinates. Daus gave also a similar analytic solution. Kaplan used a coördinate system as in I and observed that the conic passes through the four points $a_i^{-1}(a_j+a_k)^{-1/2}$ and also through the four points $[a_i(a_j+a_k)]^{-1/2}$. He would like to know if these points are of interesting geometric significance. He also stated that Ferrer's *Trilinear Coördinates* gives laborious methods for finding the foci and asymptotes; and he wishes to know if any reader can give simpler methods for this case.

The center and asymptotes of the rectangular hyperbola passing through four given points may be constructed in a way which proves some of the theorems in the above solutions and gives Steiner's generalized theorem. This construction is based on the fact that a given hyperbola of any type and its asymptotes cut from a straight line two corresponding segments with a common midpoint. It then follows that, if the hyperbola is rectangular, a right triangle

having for its hypotenuse a chord of the hyperbola and its two sides parallel to the asymptotes, is such that the straight line through the vertex of the right angle and the midpoint of the chord is a diameter of the hyperbola. Let A, B, C be the midpoints of the sides MN, NL, ML of a triangle whose vertices are on a rectangular hyperbola. On NM and ML as hypotenuses construct right triangles with sides parallel to the asymptotes, supposed known. Then the two diameters constructed as above meet in the center U of the hyperbola; and it is easily seen that angle CUA is equal to angle B of the triangle ABC, or is its supplement. Hence U is on the circumcircle of ABC, the nine-point circle of LMN. If a side is parallel to an asymptote, the hyperbola degenerates into this side and the altitude of it; but there is always at least one side not parallel to an asymptote and the construction does not fail. Conversely, if U is any point on the nine-point circle, the circle with a midpoint distinct from U, say A, as center and radius AU cuts MN in two points such that, if these are joined by straight lines with U, the two perpendiculars thus obtained cut from MN a segment with A for its midpoint. It will then be seen that the two perpendiculars cut from ML a segment with C as its midpoint. Hence the rectangular hyperbola with these lines as asymptotes passing through M and N will also pass through L. Moreover, if H is the orthocenter of LMN, the nine-point circles of LMN and HMN coincide; and, with U as center and the asymptotes determined as above, the rectangular hyperbola through M and N passes through both L and H. It now easily follows that any conic through the vertices of a triangle LMN and its orthocenter H must be a rectangular hyperbola. For the conic must be some kind of hyperbola as we see from the relative positions of the points. As shown above there exists a rectangular hyperbola through the same four points and having one asymptotic direction the same as one for the conic. Then the two conics, having five points in common, must coincide; and the given conic must be a rectangular hyperbola.

Now let X be any point chosen in the plane of L, M, N, but distinct from these points. Then the nine-point circles for LMN and XMN, if not coincident, meet in A and also in a point S. The rectangular hyperbola (S) with center S and the corresponding asymptotes, which passes through L, M, N will pass also through X. There are four such nine-point circles and they have S in common. If the two nine-point circles mentioned above coincide, then X must be at H. For, if two triangles have two vertices in common and also the same circle as nine-point circle, the two triangles either coincide or the third vertex of one is the orthocenter of the other triangle. If any two of the four nine-point circles coincide, it follows that all four coincide. If then X is not at H, the center S and the corresponding rectangular hyperbola (S) through L, M, N, X, are uniquely determined. Denote the midpoints of XL, XM, XN by L', M', N'. Then it is obvious that the points A, N', B, L', C, M' form a hexagon with its opposite sides equal and parallel. Hence a conic (K) passes through these six points with diameters AL', BN', CN' through its center K. Moreover, the circumcircles of ABC, $AM'N'$, $BN'L'$, $CL'M'$ meet in the point S. This theorem

says that, if three straight line segments AL', BM', CN', have a common midpoint K, the circumcircles of ABC, $AM'N'$, $BN'L'$, $CL'M'$ meet in a point S: a proof of this without reference to the hyperbola is simple. We now show that S is also on (K). The circle (ABC) cuts (K) in A, B, C, S'; hence BC and AS' are equally inclined in opposite senses to a principal diameter of (K). Also circle $(AM'N')$ cuts (K) in A, M', N', S'', and $M'N'$ and AS'' are equally inclined in opposite senses to the same diameter. But BC is parallel to $N'M'$, therefore AS' and AS'' are parallel; hence $S' \equiv S'' \equiv S$.

Let the feet of the altitudes of LMN be H_1, H_2, H_3, where LH_1 is one of the altitudes; then it is clear that $H_1H_2H_3$ is a self polar triangle for (S). There are three other such self polar triangles, for example, if α_1, α_2, α_3 are the feet of the altitudes of XMN, where $X\alpha\alpha_1$ is an altitude, the triangle $\alpha_1\alpha_2\alpha_3$ is self polar. Hence the intersection $\bar{L}$ of H_2H_3 and $\alpha_2\alpha_3$ is the pole of MN, and $\bar{L}M$, $\bar{L}N$ are the tangents to (S) at M and N. Also $\bar{L}A$ is a diameter; and, if $\overline{M}B$ is a diameter obtained in the same way, the intersection of $\bar{L}A$ and $\overline{M}B$ is S.

For an arbitrary point X, BC is not the polar of A with respect to (S), and we now consider the important special case where X is so situated that BC is the polar of A. Let AL cut (S) again in G and BC in $\overline{A}$, then $AG/G\overline{A} = AL/\overline{A}L = 2$; and, since $A\overline{A}L$ is a median for ABC and LMN, G must be the centroid for each triangle. It follows then that ABC is self polar with respect to (S), and if we take X at G, the centroid of ABC, this result is realized. In this case $K \equiv G$; (K) becomes (G), the Steiner ellipse with center G; and this ellipse cuts the nine-point circle of LMN in the Steiner point S, where S is the center of the rectangular hyperbola through L, M, N and the centroid G to LMN. The above constructions are now simplified, since the intersections of corresponding sides of ABC and $H_1H_2H_3$ are the poles of the corresponding sides of LMN. The Steiner ellipse is easily seen to be tangent to the sides of triangle LMN at their midpoints. This ellipse appears in the solution of 3565 [1933, 372] as the ellipse of least area circumscribing the triangle ABC, also in the solution of 3718 [1936, 442] in its relation to the roots of a cubic. From the theorem of the problem results the interesting corollary:

If a rectangular hyperbola passes through the vertices of a triangle and its centroid, it passes also through the incenter and the three excenters of the triangle formed by the midpoints of the given triangle.

Solution II gives an interesting synthetic proof of this corollary.

3798 [1936, 500]. *Proposed by N. A. Court, University of Oklahoma.*

Construct a sphere belonging to a given coaxal pencil and passing through the inaccessible point of intersection of a given line with a given plane.

Note. A similar problem in plane geometry was discussed in the *Educational Times*, Reprints, vol. 5, 1904, p. 83, Q, 15401.

Solution by the proposer.

First solution. Let I be the inaccessible point common to the given line t and the given plane (P), and let A be a point on the basic circle, assumed to be

real, of the given coaxal pencil Σ. The foot P of the perpendicular AP from A upon the plane (P) lies on the sphere (M) having AI for diameter, and the same is true about the feet Q, R of the perpendiculars from A upon any two planes (Q), (R) passing through t. The sphere $(M)\equiv APQR$ is therefore determined, hence also its center M.

Thus the midpoint M of the segment AI is known. The plane perpendicular to AM at M meets the line of centers of Σ in the center of the required sphere.

Second solution. The above solution breaks down, if Σ is a non-intersecting pencil. The solution which follows is applicable whatever the nature of the pencil Σ may be.

The spheres of Σ determine on the line t pairs of points in involution. This involution is projected from a line u taken in the plane (P) by an involution of planes (u). If I' is the trace on t of the plane (P') which corresponds to (P) in the involution (u), the sphere passing through I' and belonging to the coaxial pencil Σ solves the problem (Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 178, art. 559. Macmillan, 1935).

Editorial Note. It is not clear as to the definition of the accessible region. O.D.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A testimonial dinner was recently given by alumni of the University of Michigan to Dr. J. W. Glover, James Olney professor of mathematics and former chairman of the department, who retired from active teaching at the end of the academic year. In his honor the James W. Glover scholarship fund, made possible by contributions from his former students, was announced, and a check for \$800 as the nucleus of this fund was presented to the University. Dr. Glover was presented with a volume containing many letters expressing admiration and affection.

On May 20, 1938, at the dedication of the Franklin Institute, the University of Pennsylvania conferred an honorary degree upon Dean G. D. Birkhoff of Harvard University. At the annual meeting of the American Academy of Arts and Sciences held on May 11 in Boston, Dean Birkhoff spoke on *Mathematical advances since 1900 and their influence on scientific thought.*

Professor M. H. Stone of Harvard University has been elected a member of the National Academy of Sciences.

Professor Solomon Lefschetz of Princeton University addressed the British Association for the Advancement of Science at its August 1938 meeting. His topic was *Fixed points of transformations.*

A Guggenheim Fellowship has been awarded to Assistant Professor D. H.

Lehmer of Lehigh University. Professor Lehmer will work at various English universities on the analytic theory of numbers.

The William Lowell Putnam Prize Scholarship for 1938 has been awarded to Mr. I. Kaplansky of the University of Toronto. This scholarship is awarded annually by the Division of Mathematics at Harvard University for study at that university to one of the first five contestants in the William Lowell Putnam Mathematical Competition. Mr. Kaplansky plans to use this award during the academic year 1939–40.

Professor J. W. Lasley, Jr. of the University of North Carolina has been serving as president of the North Carolina Academy of Science for the year 1938.

Assistant Professor R. D. Agnew of Cornell University has been promoted to a professorship.

Assistant Professor L. A. Aroian is on leave of absence from Colorado State College, Fort Collins, for 1938–39 and holds a fellowship for graduate study at the University of Michigan.

Professor Emil Artin, who came to the University of Notre Dame from the University of Hamburg in the fall of 1937, has been appointed professor of mathematics at Indiana University.

Dr. Theodore Bennett of Marietta College has been promoted to an assistant professorship.

Dr. M. T. Bird of Utah State Agricultural College has been promoted to an ssistant professorship.

Professor J. W. Bradshaw of the University of Michigan has leave of absence for the first semester of the year 1938–39.

At the North Carolina State College of Agriculture and Engineering, Assistant Professor J. W. Cell has been promoted to an associate professorship.

Assistant Professor R. V. Churchill of the University of Michigan has been promoted to an associate professorship.

Associate Professor H. B. Curry of Pennsylvania State College is on leave of absence for 1938–39 and will be at the Institute for Advanced Study.

At Indiana University Professor S. C. Davisson has retired after forty-eight years of service in the department of mathematics.

Associate Professor R. D. Douglass of Massachusetts Institute of Technology has been promoted to a professorship.

Dr. W. H. Durfee, professor of mathematics at Hobart College, has been made dean of the college.

Dr. A. L. Foster of the University of California has been promoted to an assistant professorship.

Dr. Gordon Fuller of New Mexico State College has been appointed an associate professor at Alabama Polytechnic Institute.

Dr. D. W. Hall has been appointed a National Research Fellow for 1938–39, and will study at the University of Pennsylvania.

Associate Professor Frank Irwin of the University of California has been given the title emeritus.

Dr. B. F. Kimball has been appointed engineering mathematician, Depreciation Unit of the Division of Research and Valuation of the State Public Service Commission, New York, N.Y.

Assistant Professor J. H. Kusner of the University of Florida has been promoted to an associate professorship.

Dr. Saunders MacLane of the University of Chicago has been appointed an assistant professor at Harvard University.

Dr. H. M. MacNeille of Harvard University has been appointed associate professor at Kenyon College.

Dr. W. T. Martin of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Professor Karl Menger of the University of Notre Dame was visiting professor during the summer quarter at the University of California.

Assistant Professor A. B. Mewborn of the University of Arizona is on leave of absence for 1938–39.

Dr. F. J. Murray of Columbia University has been promoted to an assistant professorship.

Dr. S. B. Myers of the University of Michigan has been promoted to an assistant professorship.

E. A. Nordhaus is on leave of absence from the University of Wisconsin Extension Division and holds a fellowship for graduate study at the University of Chicago.

Dr. B. J. Pettis of the University of Virginia has been appointed a Sterling Fellow at Yale University.

Associate Professor C. H. Rawlins, Jr. of the Postgraduate School, U. S. Naval Academy, has been promoted to a professorship.

Dr. Evelyn Carroll Rusk, professor of mathematics at Wells College, has been appointed dean of the college.

Professor Hazel E. Schoonmaker of Hartwick College was married on July 14, 1938 to Professor L. T. Wilson of the United States Naval Academy.

Dr. Ruth G. Simond of the University of Michigan has been appointed to an assistant professorship at Hampton Institute.

Professor E. R. Sleight of Albion College is on leave of absence for the first semester 1938–39. He has been traveling in Norway and Sweden and expects to study in English and Scottish universities.

Assistant Professor C. H. Smiley of Brown University has been made chairman of the department of astronomy and director of Ladd Observatory, with the rank of associate professor.

Professor Virgil Snyder of Cornell University has retired with the title professor emeritus.

Assistant Professor A. W. Tucker of Princeton University has been promoted to an associate professorship.

C. B. Tucker of Kansas State Teachers College, Emporia, has been promoted to an assistant professorship.

Dr. R. J. Walker of Cornell University has been promoted to an assistant professorship.

Professor Agnes E. Wells, who has been dean of women in addition to teaching mathematics at Indiana University, has given up the work of dean to give full time to teaching.

Associate Professor W. M. Whyburn of the University of California at Los Angeles has been promoted to a professorship.

Professor K. P. Williams of Indiana University has been appointed chairman of the department of mathematics.

Associate Professor W. L. Williams of the University of South Carolina has been promoted to a professorship.

The following appointments to instructorships have been announced:
Agricultural and Mechanical College of Texas: R. R. Lyle
Bryn Mawr College: M. P. Fobes
Case School of Applied Science: Dr. J. M. Dobbie
University of California at Los Angeles: Dr. A. E. Taylor
Georgia School of Technology: Dr. G. B. Lang
Harvard University: Dr. L. A. Pipes
Haverford College: Dr. C. B. Allendoerfer
University of Illinois: Dr. P. T. Maker, Dr. E. L. Welker
Lehigh University: Dr. A. E. Pitcher, Dr. M. F. Smiley

Los Angeles City College: Dr. D. C. Duncan
Michigan College of Mining and Technology: V. O. York
University of Michigan: Dr. R. F. C. Bartels, Dr. P. C. Hammer, Dr. C. J. Nesbit
Middlebury College: L. B. Hedge
University of Nevada: Ingo Maddaus
North Park Junior College, Chicago: C. G. Erickson
University of Notre Dame: Dr. P. M. Pepper
Princeton University: W. C. Strodt
Purdue University, Dr. D. R. Shreve, Dr. M. S. Webster
Rose Polytechnic Institute: T. P. Palmer
University of Texas: Dr. H. S. Kaltenborn
United States Naval Academy: Dr. H. C. Ayres
Virginia Military Institute: I. G. Foster
Washburn College: Paul Eberhart
Wellesley College: Melita A. Holly
Wesleyan University: Dr. J. W. Wrench, Jr.
Yale University: Dr. D. C. Murdoch

Emeritus Professor E. W. Brown of Yale University died on July 22, 1938, at the age of seventy-one years. He was a charter member of the Mathematical Association.

Professor Carl Gundersen of Oklahoma Agricultural and Mechanical College died April 11, 1938. He was a charter member of the Mathematical Association.

Doctor D. N. Lehmer, emeritus professor of mathematics at the University of California, died September 8, 1938. He was a charter member of the Mathematical Association and had served both as vice-president and as trustee.

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